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Essays in Economics and Econometrics

ESSAYS IN ECONOMICS AND ECONOMETRICS

A Volume in Honor of Harold Hotelling

Edited by

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Harold Hotelling

Harold Hotelling was born in Fulda, Minn., on September 29, 1895. While he was still a small boy his family moved to Seattle, Wash., where he spent most of his youth.

After finishing high school in Seattle he worked on small weekly newspapers in Washington from 1915 to 1916. He was graduated from the University of Washington in 1919 and took an M. A. degree from the same institution in 1921. He earned his doctorate in mathematics at Princeton University in 1924. As a graduate student he taught mathematics at both Washington and Princeton.

His first permanent academic position was that of mathematical consultant at the Stanford Food Research Institute, a position he held from 1924 until 1927. He became associate professor of mathematics at Stanford University in 1927 and continued in this position until 1931. In 1929 he spent several months at the Rothamsted Experiment Station, working with Sir Ronald Fisher.

In 1931 he left Stanford University to become professor of economics at Columbia University. During his years at Columbia he served as head of the Statistical Research Group of the Division of War Research, and was instrumental in the founding of the Department of Mathematical Statistics at Columbia.

In 1946 he accepted his present position as Professor of Mathematical Statistics and Associate Director of the Institute of Statistics at the University of North Carolina. He was subsequently given the additional post of Professor of Economics, and in this capacity he teaches courses in mathematical economics.

Many honors and recognitions have been awarded him. He is past president of the Econometric Society, the Indian Statistical

Congress, and the Institute of Mathematical Statistics. He served as vice president of the American Statistical Association and Section K of the American Association for the Advancement of Science. He was awarded an honorary Doctor of Laws degree by the University of Chicago in 1955.

A bare recital of biographical facts does little more than hint at the important contributions that Professor Hotelling has made to the mathematical and social sciences. Few men can match his mastery of mathematics, statistics, and economics, and fewer yet can match his record of major contributions in all three fields. In the field of economics, his contributions to the theory of utility and demand, welfare economics, and the incidence of taxation have already assumed the stature of classics. In addition, his contributions to statistics have found applications in econometrics as well as in other areas of the physical, biological, and social sciences.

Professor Hotelling's pioneering work in the teaching of mathematical statistics and mathematical economics was fundamental to achieving recognition for these important disciplines in American universities. Here one may cite the important part he played in developing these areas at Columbia University and the University of North Carolina. The generations of graduate students who have worked with him also indicate the major role he has played in developing mathematical statistics and mathematical economics in the United States.

The essays in this volume, dealing with topics in mathematical economics and econometrics, are offered by the authors as indications of their respect for the contributions that Harold Hotelling has made to economics. The authors, some of them

former students and all of them long-time associates of Professor Hotelling, also offer these essays as tokens of their warm personal regard for him.

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PART I
On the Mathematics of Optimization

PAUL A. SAMUELSON

Structure of a Minimum Equilibrium System

1. The functions met with in classical thermodynamics possess an abstract structure shared by equilibrium systems in quite diverse other fields — e.g., statics, economic theory, the recently developed mathematical theory of games and linear programming, etc.[1]) Given a set of numerical data on punch cards purporting to come from an equilibrium system sharing these properties, we should be able to determine whether such a supposition is tenable. Once the full empirical implications of the hypotheses embodied in the equilibrium theories have been elucidated, we can, if we wish, dispense with that theory except for mnemonic purposes.

By use of matrix notation, the present paper gives a brief but exhaustive summary of the properties of such systems, confining itself for simplicity to the *regular* case of unique and differentiable functions. Then it develops an exhaustive set of conditions that any finite set of observations must meet if the theory in question is not to be refuted.

2. Imagine that we are given a set of $2n$ variables that come in conjugate pairs (x_i, y_i). (In phenomenological thermodynamics these might be designated as generalized coordinates and generalized forces, such as entropy and temperature, volume and negative pressure, chemical mass and chemical potential, etc.)[2])

[1]) For the thermodynamic and economic interpretation of the mathematical relations here analyzed see such references as E. A. Guggenheim, *Thermodynamics* (Amsterdam, 1950); L. Tisza, "On the General Theory of Phase Transitions," in *Phase Transformations in Solids* (Wiley, 1951, N.Y.), pp. 1—37; P. A. Samuelson, *Foundations of Economic Analysis* (Cambridge, Mass. 1947), Ch. 3.

[2]) To avoid indeterminacy of scale, all extensive quantities can be thought of as expressed in ratio to the amount of an extensive variable, x_{n+1}, which is not included in this analysis.

There is assumed to exist n equilibrium conditions or equations of state between these $2n$ variables, defined as follows:

Axiom for a Regular Equilibrium System: Corresponding to prescribed values for n variables $(y_1, \ldots, y_n)$, values of the conjugate variables $(x_1, \ldots, x_n)$ are determined so as to provide a regular minimum to $F(x; y) = F(x_1, \ldots, x_n; y_1, \ldots, y_n) = Y(x_1, \ldots, x_n) - \sum_1^n y_j x_j$, so that

$$(1)\quad \partial F/\partial x_i = 0 = Y_i(x_1, \ldots, x_n) - y_i \quad (i = 1, \ldots, n)$$

where subscripts to functions always stand for partial derivatives with respect to the indicated variables, and where the Hessian matrix of second derivatives of F and Y with respect to the x's is everywhere defined and is positive definite so as to insure a regular minimum.

Knowledge of the function Y above determines all properties of the equilibrium system. (In thermodynamics, Y is internal energy.) These regularity conditions of smoothness and convexity could be relaxed. But for the present it is sufficient to deduce the full implications of the regular case. We have immediately from the Axiom:

Theorem 1: Functions constraining a set of variables are equivalent to a regular equilibrium system, if and only if, they can be thrown into the form:

$$(2)\quad x_i = X^i(y_1, \ldots, y_n) \equiv X_i(y_1, \ldots, y_n) \quad (i = 1, \ldots, n),$$

where $X(y_1, \ldots, y_n)$ is an existent twice differentiable function, whose Hessian matrix of partial derivatives — which is also the symmetric Jacobian of the transformation (2) — must be positive definite.

The necessity part of this theorem follows immediately from the fact that the relations (1) can always be inverted by virtue of the fact that their Jacobian is postulated to be the nonsingular matrix of a positive definite quadratic form. Equations (2) represent such inverse functions, and their Jacobian $[X_j^i] = [Y_{ij}]^{-1}$. Since the inverse of a symmetric matrix must be symmetric, our right to set $X^i = X_i$ is established and the integrability or symmetry conditions guarantee the existence of the function $X(y_1, \ldots, y_n) = \Sigma \int X_i dx_i$ independently of

path, which is unique up to an inessential integration constant. Finally, because the inverse of a positive definite matrix must be positive definite, this stipulated property is indeed necessary.

The sufficiency part of the theorem can be most easily seen by recognizing the dualism between the x's and y's. If we interchanged the letters in (2), we should have a relation with all the properties of (1); call it (1)* and call (2)* the relation (1) with the labels interchanged. Then by the just proved necessary condition, it follows that (1)* implies (2)*, which is all that we need for our sufficiency proof of Theorem 1. This also shows the truth of

Corollary 1: The variables $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are mutually conjugate: the same equilibrium system can be defined by the dual minimum problem.
$F^*(y; x) = X(y_1, \ldots, y_n) - \sum_1^n x_j y_j$, a minimum for prescribed x's.

3. The equilibrium conditions of (1), or (2), were seen to be n constraints among $2n$ variables. There are a vast number of ways of taking as prescribable variables n out of $2n$ variables, namely $(2n)!/(n!)^2$ permutations. But in the general case, there is no reason to believe that any n variables taken at random can be independently varied. Thus, in the simple case where $Y(x_1, x_2, \ldots) = \frac{1}{2}\sum x_j^2$, the equations of state are $x_i = y_i$, and the Hessians of Y and of X are the identity matrixes. Clearly for this system, the variables $(x_1, y_1, x_3, \ldots)$ cannot be independently varied.

In general, the only admissible sets that we can be sure of are those generated by the following rule: *Select one and only one member from each conjugate pair of variables.* Since there are two independent ways of selecting from each pair, there are obviously exactly 2^n such admissible sets of independently prescribable variables. If we follow the convention of suitably renumbering the variables, we can write any of the 2^n sets in one of the following $n + 1$ forms:

$$(y_1, \ldots, y_n),\ (y_1, \ldots, y_{n-1};\ x_n), \ldots, (y_1, \ldots, y_m;\ x_{m+1}, \ldots, x_n), \ldots, (y_1;\ x_2, \ldots, x_n),\ (x_1, \ldots, x_n).$$

The first and last of these are the "unmixed" sets of (1) and (2).

That the remaining "mixed" sets are admissible as independently prescribable follows from the fact that every principal minor of the definite Jacobians of (1) and (2) are necessarily definite and non-singular. This assures us that the first m equations of (1) can be solved explicitly for $(x_1, \ldots, x_m)$ in terms of $(y_1, \ldots, y_m;\ x_{m+1}, \ldots, x_n)$ since the crucial matrix of partial derivatives $[\partial y_i/\partial x_j]$ is the non-singular m rowed principal minor of $[Y_{ij}]$; with these x's solved for, we can substitute in the remaining $n - m$ equations of (1) to get $(y_{m+1}, \ldots, y_n)$ in terms of the same independent variables. (A dual similar argument could be made concerning the solvability for $(y_{m+1}, \ldots, y_n)$ of the last $n - m$ equations of (2).) Thus, by use of the Implicit Function Theorem — and by an extension of it that gives nowhere vanishing principal minors as sufficient conditions for uniqueness of our solutions in the large — we can deduce.

Theorem 2: For any of the 2^n admissible sets of n independent variables, consisting of one and only one member of each pair of conjugate variables, we can in regular systems — and only in such systems — determine unique and differentiable values of the remaining variables. After we have suitably renumbered the variables, we can write these relations as

$$
\begin{aligned}
&x_i = x^i(\bar{y}_1, \ldots, \bar{y}_m;\ \bar{x}_{m+1}, \ldots, \bar{x}_n) \quad (i = 1, \ldots, m) \\
&y_i = y^i(\bar{y}_1, \ldots, \bar{y}_m;\ \bar{x}_{m+1}, \ldots, \bar{x}_n) \quad (i = m + 1, \ldots, n), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leqq m \leqq n
\end{aligned} \tag{3}
$$

with the partitioned Jacobian matrix of skew-symmetric form

$$
\begin{aligned}
J_m &= \begin{bmatrix} \dfrac{\partial x}{\partial \bar{y}}, & \dfrac{\partial x}{\partial \bar{x}} \\[2ex] \dfrac{\partial y}{\partial \bar{y}}, & \dfrac{\partial y}{\partial \bar{x}} \end{bmatrix} = \begin{bmatrix} Y_{xx}^{-1}, & -Y_{xx}^{-1} Y_{x\bar{x}} \\ Y_{\bar{x}x} Y_{xx}^{-1}, & Y_{\bar{x}\bar{x}} - Y_{\bar{x}x} Y_{xx}^{-1} Y_{x\bar{x}} \end{bmatrix} \\
&= \begin{bmatrix} X_{\bar{y}\bar{y}} - X_{\bar{y}y} X_{yy}^{-1} X_{y\bar{y}}, & X_{\bar{y}y} X_{yy}^{-1} \\ -X_{yy}^{-1} X_{y\bar{y}}, & X_{yy}^{-1} \end{bmatrix} = \begin{bmatrix} Y_{xx}^{-1} & -Y_{xx}^{-1} Y_{x\bar{x}} \\ -X_{yy}^{-1} X_{y\bar{y}} & X_{yy}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} A & B \\ -B' & C \end{bmatrix}, \quad 0 \leqq m \leqq n,
\end{aligned} \tag{4}
$$

Y_{xx} being understood to stand for the first m by m rows and columns of $[Y_{ij}], \ldots, X_{\bar{y}\bar{y}}$ for the last $n - m$ rows and columns

of $[X_{ij}]$, etc.; and the symmetric diagonal blocks A and C being positive definite.[3])

In (3) the position of the semi-colon will serve to identify the value of m, except in the limiting cases where $m = 0$ or n, in which case we have the unmixed variables and equations of (1) and (2). To prove the various relations of (4), we make the appropriate matrix substitutions (or matrix premultiplications) in the differentials relation of (1) and (2)

$$\begin{bmatrix} Y_{xx} Y_{x\bar{x}} & -I_m & 0 \\ Y_{\bar{x}x} Y_{\bar{x}\bar{x}} & 0 & -I_{n-m} \end{bmatrix} \begin{bmatrix} dx \\ d\bar{x} \\ d\bar{y} \\ dy \end{bmatrix} = 0,$$

(5)

$$\begin{bmatrix} -I_m & 0 & X_{\bar{y}\bar{y}} & X_{\bar{y}y} \\ 0 & -I_{n-m} & X_{y\bar{y}} & X_{yy} \end{bmatrix} \begin{bmatrix} dx \\ d\bar{x} \\ d\bar{y} \\ dy \end{bmatrix} = 0,$$

to get the first row's relations of (4), which are seen to be dual forms. The next row of (4) immediately follows. The last form is important: it summarizes the facts that the diagonal blocks are symmetrical, but that the off-diagonal matrixes are skew-symmetric with $(\partial x/\partial \bar{x}) = -(\partial y/\partial \bar{y})'$. These properties follow from the symmetry relations $[Y_{ij}] = [Y_{ji}]$, while the definiteness of the positive blocks follows from the definiteness of $[Y_{ij}]$ and $[X_{ij}]$ and of the inverses of their principal minors.

To prove the converse part of the theorem — that the structure of (3) and (4) implies the structure of (1) or (2) — note that by the same routine substitutions by which (5) went into (4), but working in reverse, we find for the Jacobian of $(\bar{y}, y)$ in terms of $(x, \bar{x})$

[3]) If determinacy of scale has not been achieved by the device of the first footnote, then $Y(x_1, \ldots, x_n)$ will be a homogeneous function of order one, and its n^2 Hessian $[Y_{ij}]$ will be positive definite of rank $n - 1$. The functions (2) will then be determinate only in ratio and not in scale. For $m < n$, the functions (3) will be perfectly determinate, and can be shown to be homogeneous of order one in $[x_{m+1}, \ldots, x_n]$ alone, requiring C to be positive semi-definite of rank $(n - m - 1)$ with $[x_{m+1}, \ldots, x_n]\, C = 0$ and $[x_1, \ldots, x_n] = [x_{m+1}, \ldots, x_n] B'$.

$$
(6) \qquad J_n = \begin{bmatrix} \dfrac{\partial \bar{y}}{\partial x}, \dfrac{\partial \bar{y}}{\partial \bar{x}} \\[2ex] \dfrac{\partial y}{\partial x}, \dfrac{\partial y}{\partial \bar{x}} \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ -B'A^{-1} & C+B'A^{-1}B \end{bmatrix}
$$

This is seen to be symmetric. It remains to show that independently of B, the positive definiteness of C and A will guarantee that this whole matrix is positive definite, and hence of the stipulated regular form $[Y_{ij}]$.

We make use of the easily demonstrable algebraic fact that any positive definite A^{-1} can be written in an infinity of different ways as $D'D$, where D is non-singular.

Now consider the quadratic form

$$
\begin{aligned}
Q = Q_1 + Q_2 &= [z_1, \ldots, z_m, z_{m+1}, \ldots, z_n] J_n [z_1, \ldots, z_m, z_{m+1}, \ldots, z_n]' \\
&= [z_{m+1}, \ldots, z_n] C [z_{m+1}, \ldots, z_n]' \\
&+ [z_1, \ldots, z_n] \begin{bmatrix} D'D, & -D'(DB) \\ -(DB)'D, & (DB)'(DB) \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \\
&= [0, \ldots, 0, z_{m+1}, \ldots, z_n]\, C [0, \ldots, 0, z_{m+1}, \ldots, z_n]' + WW'
\end{aligned}
$$

where $W = [z_1, \ldots, z_m]D' - [z_{m+1}, \ldots, z_n](DB)'$.

Q_1 is positive semi-definite because C was positive definite by hypothesis; Q_2 is a sum of squares and hence positive semi-definite. But for $z'z \neq 0$, where $Q_1 = 0$, $Q_2 > 0$ and where $Q_2 = 0$, $Q_1 > 0$; hence, Q is finally proved to be positive definite.

4. Returning now to the mixed relations (3), their skew-symmetry property warns us that, as they stand, these functions are not partial derivatives of any existent parent function. However, because of the special skew-symmetry between the off-diagonal blocks, it is clear that changing the algebraic sign of the two upper blocks will give us a symmetric matrix

$$
(7) \qquad \begin{bmatrix} -A & -B \\ -B' & C \end{bmatrix}
$$

which can be regarded as the second-derivative Hessian matrix of an existent function for each m. Hence

Theorem 3: Any regular equilibrium system in $2n$ conjugate variables has associated with it, for each of the 2^n admissible sets of independent variables, a characterizing function, which

may be written

$$(8)\quad \begin{aligned} P(y_1, \ldots, y_m;\ x_{m+1}, \ldots, x_n) = \sum_1^m \int -x^i(y_1, \ldots, y_m; \ldots, x_n)\,dy_i \\ + \sum_{m+1}^n \int y^i(y_1, \ldots, y_m; \ldots, x_n)\,dx_i \end{aligned}$$

where the line integral is independent of path by virtue of the symmetry relations. In terms of the relevant characterizing function, we can rewrite equations (3)

$$(3)'\quad \begin{aligned} x_i &= -\partial P(y_1, \ldots, y_m;\ \ldots, x_n)/\partial y_i, \quad i = 1, \ldots, m \\ y_i &= +\partial P(y_1, \ldots, y_m;\ \ldots, x_n)/\partial x_i, \quad i = m+1, \ldots, n. \end{aligned}$$

Corollary 1: For $m = 0$, $P \equiv Y$; and for $m = n$, $P \equiv -X$; for all m and for values of the $2n$ variables satisfying the n equilibrium conditions,

$$(9)\quad P(y_1, \ldots, y_m;\ x_{m+1}, \ldots, x_n) = Y - \sum_1^m y_j x_j = -X + \sum_{m+1}^n y_j x_j;$$

but these are not identities in the $2n$ variable space.

Corollary 2: For prescribed values of $(x_1, \ldots, x_m; y_{m+1}, \ldots, y_n)$ the expression

$$(10)\quad F^m(x, \bar{x}; \bar{y}, y) = P(\bar{y}_1, \ldots, \bar{y}_m;\ \bar{x}_{m+1} \ldots, \bar{x}_n) + \sum_1^m x_i y_i - \sum_{m+1}^n x_i y_i$$

reaches a regular saddlepoint or minimax when, and only when, the remaining variables $(\bar{y}_1, \ldots, \bar{y}_m; \bar{x}_{m+1}, \ldots, \bar{x}_n)$ are in their regular equilibrium configuration; i.e., for an equilibrium position $(x_1^0, \ldots; \ldots, \bar{x}_n^0;\ \bar{y}_1^0, \ldots; \ldots y_n^0)$

$$(11)\quad F^m(x^0, \bar{x}^0; \bar{y}, y^0) \leqq F^m(x^0, \bar{x}^0;\ \bar{y}^0, y^0) \leqq F^m(x^0, \bar{x}; \bar{y}^0, y^0)$$

with the inequalities valid for distinct points.

In classical thermodynamics these 2^n characterizing functions have a variety of notational descriptions and go by such names as internal energy, enthropy, enthalpy, Helmholtz free energy, Gibbs potential or free energy, etc. Knowledge of any one of them provides a succinct summary of all the regular equilibrium relations, and by use of the transformations of (6) and (4) we can in principle go from the properties of any one to the properties of any other.

The symmetry properties of (4) provides immediate proof of the existence of each of these characterizing functions described in theorem 3. To prove Corollary 1, we may use the properties of Legendre transformations or of differential geometry; but remaining within the limits of the present straightforward algebraic manipulation of Jacobian matrices, we make the equilibrium substitutions to get

$$Y-\sum_1^m y_j x_j = Y(x^1(\bar{y}_1,\ldots;\ldots,\bar{x}_n),\ldots;\ldots,\bar{x}_n)-\sum_1^m \bar{y}_i x^i(\bar{y}_1,\ldots;\ldots,\bar{x}_n)$$

and

$$(12)\quad \begin{aligned} \frac{\partial(Y-\sum_1^m y_j x_j)}{\partial \bar{y}_i} &= \sum_1^m (Y_j-\bar{y}_j)x_i^j - x_i = 0-x_i, \quad (i=1,\ldots,m) \\ \frac{\partial(Y-\sum_1^m y_j x_j)}{\partial \bar{x}_i} &= \sum_1^m (Y_j-\bar{y}_j)x_i^j + Y_i = 0+y_i, \; (i=m+1,\ldots,n) \end{aligned}$$

which is seen to be identical with P if the inessential integration constant is appropriately adjusted. By using the dualism between Y and X, the relations of P to $-(X-\sum_{m+1}^n x_i y_i)$ shown in (9) are verified.

To prove the saddlepoint property of Corollary 2, we note the equivalence of

$$(13)\quad \frac{\partial F^m}{\partial \bar{y}_i}=0 \quad (i=1,\ldots,m); \quad \frac{\partial F^m}{\partial \bar{x}_i}=0 \quad (i=m+1,\ldots,n)$$

to (3)′; and we note that in changing the algebraic signs in the upper blocks of (4), we achieve (7)'s symmetry but at the cost of making its upper diagonal block $-A=-Y_{xx}^{-1}$ be negative rather than positive definite. Together with the positive definiteness of $C=X_{yy}^{-1}$, this is sufficient for the saddle-point or minimax property.

This minimax property shows us that the notion of an unmixed set of conjugate variables $(y_1,\ldots,y_n)$ and $(x_1,\ldots,x_n)$ is not arbitrary. We are generally forced to group y_1 and y_2 together and are not free to interchange the x and y designations of y_1 and x_1. (Thus, entropy and volume belong together in a sense that temperature and volume do not; this we know from

the mathematics as well as from the physics.) For if we link up admissible sets of conjugate variables without distinguishing which are x's and which are y's, the Jacobian (4) of the resulting transformation (3) will have a *determinable* number of rows and columns that yield skew-symmetry. Thus, m is a determinable number which can be verified to be 0 or n or to be in between. This enables us to find the proper way of dividing the $2n$ conjugate variables into two unmixed groups. Of course, which we call x and y has no mathematical significance because of the complete dualism between them. (In the singular case where whole rows and corresponding columns of J_m vanish, we shall later show that the exact size of m is indeterminable but indifferent, since the system then splits off into independent constellations.)

5. Theorem 3 and its corollaries each contains within itself, and is equivalent to, the previous theorems and Axiom. If any $2n$ variables can be labelled so that n of them can be expressed in terms of the remaining n by a transformation with the regular minimax properties of (11), such a system must have all the properties of a regular equilibrium system.

For each of the 2^n admissible sets of independent variables, there will be $n(n-1)/2$ symmetry or skew-symmetry relations. In thermodynamics these are called Maxwell relations, and any one complete set of them implies all the rest. Matrix notation and operations themselves constitute a most excellent mnemonic device; so the transformation of (4) and (6) represent an excellent substitute for the Bridgman[4]) and similar formulas by which partial derivatives of one set can be converted into partial derivatives of another. Given numerical coefficients, we can efficiently go from, say, J_6 to J_9, by the series of steps J_6, J_7, J_8, J_9, where at each stage the only needed operations are of the Gauss-Doolittle type convenient for desk and other calculators, namely $a - bc$, and $(a - bc)/d$. In practice, there is never need actually to renumber variables since the four partitioned matrices can always be represented by different colors for the numbers or by appropriate underlining.

[4]) P. W. Bridgman, A Condensed Collection of Thermodynamic Formulas (Harvard, 1925).

We might call a system a "regular variational system" if it satisfies all the symmetry equalities of a regular minimum system, whether or not it also has the definiteness properties on its Jacobians of a minimum. Before exploring further the inequalities implied by the minimum property itself, we first give a single minimum formulation that is capable of handling all admissible mixed cases simultaneously.

6. Our original axiom assures us that $F(x; y)$ attains a minimum if we set the x's at their optimal equilibrium values in terms of the y's as determined by the equilibrium relations (1) or (2). Thus, the minimum obtainable of F will depend only on the y's and we may write it as $F(y) \leqq F(x; y)$, with the equality sign holding in equilibrium, and only then. It will be easily shown that $F(y) \equiv -X(y)$ for an appropriate choice of integration constant in the latter dual function to $Y(x)$. This suggests

Theorem 4: For any regular equilibrium system, we can find two mutually conjugate functions $Y(x_1, \ldots, x_n)$ and $X(y_1, \ldots, y_n)$, such that

$$(14) \quad N(x_1, \ldots, x_n; y_1, \ldots, y_n) = Y(x_1, \ldots, x_n) + X(y_1, \ldots, y_n) - \sum_1^n x_i y_i$$

reaches a regular minimum value of zero if, and only if, the $2n$ variables are in their n equilibrium relations to each other. The Hessian matrices of Y and X will be positive definite everywhere and in the equilibrium configurations will be exact inverses of each other.

Note that this minimum formulation includes as special cases the dual minimum problems of the Axiom and Corollary 1. Note that it is the only formulation yet stated that treats both sides of the dual with perfect symmetry, and the only minimum formulation that handles all admissible mixed sets of variables. If the Y, X, and P functions are called "potentials," then $N(x; y)$, which is defined in the $2n$ space and not alone on the n-dimensional equilibrium locus, is in the nature of "superfluous or disequilibrium potential."

To prove theorem (4), we note that by $F(y)$'s definition,

$F(x; y) - F(y)$ does have the stipulated properties of $N(x; y)$. It remains to show that $X(y)$, as previously defined from (2), with its Hessian $[X_{ij}] = [Y_{ij}]^{-1}$, is for an appropriate choice of integration constant identical to $-F(y)$. But note that for $m = 0$, $P(y_1, \ldots, y_n)$, as defined already in (8), is identically $F(y_1, \ldots, y_n)$; and as shown in (9),

$$\frac{\partial F}{\partial y_i} = \frac{\partial P(y_1, \ldots, y_n)}{\partial y_i} = -x_i \quad (i = 1, \ldots, n)$$

which agrees with (2)'s definition of X. Hence, $X(y_1, \ldots, y_n) = -F(y_1, \ldots, y_n)$.

Corollary 1: The regular equilibrium is defined in terms of any admissible set of mixed variables by

$$(15) \qquad \begin{aligned} \frac{\partial N}{\partial x_i} &= 0 = Y_i(x_1, \ldots, x_n) - y_i \quad (i = 1, \ldots, n) \\ \frac{\partial N}{\partial y_i} &= 0 = X_i(y_1, \ldots, y_n) - x_i \quad (i = m + 1, \ldots, n) \end{aligned}$$

where the Hessian of N with respect to $(x_1, \ldots, x_m; y_{m+1}, \ldots, y_n)$ is positive definite being of the form

$$(16) \qquad \begin{bmatrix} Y_{xx}^{-1} & 0 \\ 0 & X_{yy}^{-1} \end{bmatrix}$$

If we were to prescribe an inadmissible set of variables, involving both an x_i and y_i, then we would get -1 terms in the off-diagonal blocks of the Hessian matrix (16), and we could not infer that it has the positive definiteness properties needed for a minimum. Thus, in the case earlier mentioned where $x_i = y_i$, $N(x; y) = \frac{1}{2}\sum x_j^2 - \sum x_j y_j = \frac{1}{2}\sum(x_j - y_j)^2 + \frac{1}{2}\sum y_j^2$ and prescribing $(x_1, y_1, y_3, \ldots, y_n)$ and minimizing with respect to the remaining variables would give $N = \frac{1}{2}(x_1 - y_1)^2$, which need not vanish. This confirms that only 2^n sets of variables are generally independently variable.

It will be noted that (15) shows the first m equations of (1) and the last $(n - m)$ equations of (2) together to be equivalent to (1), or to (2), or to (3). Transforming the differentials of (15) by the usual methods will give us J_m in the next to the last form

of (4), a form that treats both aspects of the dual with perfect symmetry. It will be noted that the P functions defined in (9) did not treat the dual functions Y and X perfectly symmetrically, there being a difference in algebraic sign. But we can split N up into two parts, so that

$$
\begin{aligned}
N(x;y) &= (Y - \sum_{1}^{m} y_j x_j) - (-X + \sum_{m+1}^{n} y_j x_j) \\
&= -(Y - \sum_{1}^{m} y_j x_j) - (X - \sum_{m+1}^{n} y_j x_j) \qquad (17) \\
\text{(in equil.)} &= P(y_1, \ldots, y_m; \ldots, x_n) - P(y_1, \ldots, y_m; \ldots, x_n) \\
&= P(y_1, \ldots, y_m; \ldots, x_n) + P^*(x_1, \ldots, x_m; \ldots, y_n)
\end{aligned}
$$

where the definition of the functions P^*, mutually complementary to P, is obvious. Through the whole $2n$ space, P and P^* are independent of each other, even though they are complementary on the n-dimensional equilibrium locus.

The minimum formulation in terms of N immediately confirms that a set like $(y_1, \ldots, y_n; x_{m+1}, \ldots, x_n)$ is truly a mixed one, since generally $N - \sum_1^n x_j y_j$ can be resolved into two additive functions of n variables each in essentially one way only, as can be determined from the pattern of zeros in the $2n$ by $2n$ Hessian matrix of N. The singular case mentioned earlier, where $Y(x)$ and $X(y)$ can be each written as the sum of two functions involving no overlapping variables, is instantly revealed by the use of $N(x; y)$. Thus, if $Y(x_1, \ldots, x_n) = R(x_1, \ldots, x_r) + S(x_{r+1}, \ldots, x_n)$, X will be capable of a similar split and we are really dealing with two entirely independent sub-systems. That being the case, there is no mathematical way of relating x_1 to x_n in any sense different from the relating of x_1 to y_n.

Finally, it may be mentioned that the general minimum formulation in terms of N remains valid in irregular cases where some of the functions have corners with undefined partial derivatives, and where the equilibrium relations may not be unique or continuous. (Even the simplest case of phase equilibrium of a solid and liquid provides an "irregular" example where functions have a corner and partial derivatives are not defined.)

7. This exhausts the full empirical implications upon our observable functions. If someone specifies in detail any given

set of functions, we can in principle compute various sets of partial derivatives, test various of the symmetry relations, and various of the determinantal conditions for definiteness. However, suppose that the functions specified are purely empirical and not exactly represented by any finite combination of known mathematical formulas. We have a procedure for calculating the values of the equilibrium functions at any point. Also, we can approximate to various of their derivatives; however, unless we are given certain *a priori* bounds on the higher derivatives of the function, no matter how far we push our computations, we cannot strictly infer the goodness of our approximations and therefore cannot, strictly speaking, ever test a symmetry relation with complete rigor.

8. Fortunately, in applied sciences, we usually have at least vague notions concerning smoothness of higher derivatives, and we can therefore compute an expression like

$$\left\{\frac{Y^2(x_1+h_1, x_2, \ldots)-Y^2(x_1, x_2, \ldots)}{h_1} - \frac{Y^1(x_1, x_2+h_2, \ldots)-Y^1(x_1, x_2, \ldots)}{h_2}\right\} \tag{18}$$

for "small" h_1 and h_2, and decide whether it is sufficiently far from zero to refute the hypothesis that $\partial Y^2/\partial x_1 = \partial Y^1/\partial x_2$.

9. J. R. Hicks[5]) has derived an interesting generalization of the above finite tests for integrability or symmetry. For this purpose, let us suppose that the $(Y^1, \ldots, Y^n)$ functions of the x's are indeed the partial derivatives of a function $Y(x_1, \ldots, x_n)$, which has continuous second partial derivatives. Then we can write various Taylor's expansions such as

$$\begin{aligned} Y(x_1^1, x_2^1, \ldots) - Y(x_1^2, x_2^2, \ldots) &= \sum_1^n (x_j^1 - x_j^2) Y_j(\bar{x}_1, \bar{x}_2, \ldots) + 0 \\ &= \sum_1^n (x_j^1 - x_j^2) Y_j(x_1^2, x_2^2, \ldots) + R_2' \\ &= \sum_1^n (x_j^1 - x_j^2) Y_j(x_1^1, x_2^1, \ldots) - R_2'' \qquad (19) \\ &= \sum_1^n (x_j^1 - x_j^2) \frac{Y_j(x_1^1, x_2^1, \ldots) + Y_j(x_1^2, x_2^2, \ldots)}{2} + R_3 \end{aligned}$$

[5]) J. R. Hicks, *A Revision of Demand Theory* (Oxford, 1956), p. 126.

where $(\bar{x}_1, \bar{x}_2, \ldots)$ or $(\bar{x})$ represents a point intermediate between (x^1) and (x^2); where R_2' and R_2'' represent remainder terms involving second-degree or higher terms in $(x_j^1 - x_j^2)$; and where R_3 involves third-degree or higher terms in $(x_j^1 - x_j^2)$ because R_2' and $-R_2''$ can be shown to differ by terms of higher than the second degree.

Following Hicks, for the rest of this section I omit all terms of higher than second degree and state relations that the resulting approximations must satisfy. Of course

$$\begin{aligned} 0 &= [Y(x^1) - Y(x^2)] + [Y(x^2) - Y(x^3)] + [Y(x^3) - Y(x^1)] \\ &= \tfrac{1}{2}\sum(x_j^1 - x_j^2)(y_j^1 + y_j^2) + \tfrac{1}{2}(x_j^2 - x_j^3)(y_j^2 + y_j^3) + \tfrac{1}{2}\sum(x_j^3 - x_j^1)(y_j^3 + y_j^1) \\ (20) \quad &= \tfrac{1}{2}\sum(x_j^1 y_j^2 - \sum x_j^2 y_j^1) + \tfrac{1}{2}(\sum x_j^2 y_j^3 - \sum x_j^3 y_j^2) + \tfrac{1}{2}(\sum x_j^3 y_j^1 - \sum x_j^3 y_j^1) \end{aligned}$$

where the equilibrium substitutions $y_i = Y_i(x_1, \ldots, x_n)$ have been made.

This is a generalized integrability condition that any three near-by points must approximate to. This general Hicks condition includes as a special case the above relation (18); to see this set $(x^1) = (x_1, x_2, \ldots)$, $(x^2) = (x_1 + h_1, x_2, \ldots)$, $(x^3) = (x_1, x_2 + h_2, \ldots)$, and rearrange the resulting terms of (20) to get (18).

An obvious slight generalization of the Hicks reasoning will give us conditions that any 4, 5, 6, . . ., or r distinct points must satisfy as they come close together.

Theorem 5: Any r nearby points observed from a regular, continuously smooth, minimum system must satisfy the finite equality

$$\begin{aligned} 0 &= \sum_{k=1}^{r} (Y(x^k) - Y(x^{k+1})) + (Y(x^r) - Y(x^1)) \\ (21) \quad &= \sum_{k=1}^{r} \left(\tfrac{1}{2}\sum_{j=1}^{n}(x_j^k - x_j^{k+1})(y_j^k + y_j^{k+1})\right) + \tfrac{1}{2}\sum_{j=1}^{n}(x_j^r - x_j^1)(y_j^r + y_j^1) + R_3 \\ &= \tfrac{1}{2}\Big((\sum_1^n x_j^1 y_j^2 - \sum x_j^2 y_j^1) + (\sum_1^n x_j^2 y_j^3 - \sum x^3 j^2) + \ldots \\ &\qquad + (\sum_1^n x_j^r y_j^1 - \sum x_j^1 y_j^r)\Big) + R_3 \end{aligned}$$

where R_3 involves terms of at least the third degree in $(x_j^i - x_j^k)$. By setting (x^1), (x^2), (x^3), (x^4) equal to $(x_1, x_2, \ldots)$,

$(x_1 + h_1, x_2, \ldots)$, $(x_1, x_2 + h_2, \ldots)$, $(x_1 + h_1'', x_2, \ldots)$, we can convert the above into the equality that must hold in the limit between the various more familiar finite approximations to

$$\frac{\partial^3 Y}{\partial x_1^2 \partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right) = \frac{\partial}{\partial x_2}\left(\frac{\partial^2 Y}{\partial x_1^2}\right), \text{ etc.};$$

and still other special cases can be derived involving chains of any number of observations in excess of 2.

10. Equations (21) would be adequate tests for symmetry and for the variational (as distinct from minimum) aspect of the problem were it not for the fundamental methodological difficulty already mentioned in paragraph 7. Even if we could exactly calculate — free of all statistical or experimental error — any point on the n-dimensional equilibrium locus in the $2n$-dimensional space, nonetheless we could never be perfectly certain that an observed value for the bracketed expression in (18) did or did not refute integrability. If the bracketed test expression exactly equals zero, we cannot be sure that for still closer together points it will not turn out to differ from zero. If the test differs from zero, we cannot be sure that the difference is greater than the admissible remainder term R_3. Pragmatically, we may sidestep this methodological difficulty by hypothesizing *a priori* knowledge; or we may not care whether the equilibrium system is "really" variational, provided it behaves sufficiently like one for the purpose at hand.

The advantage of being able to convert a problem into variational form seems to be mainly mnemonic. Instead of having to record or remember n independent functions $Y_i(x_1, \ldots, x_n)$, we need only to know one function $Y(x_1, \ldots, x_n)$. Or it is enough to know everywhere the one function $\partial Y(x_1, \ldots, x_n)/\partial x_1$, with $\partial Y(\bar{x}_1, x_2, \ldots, x_n)/\partial x_2$ being only known everywhere in the subspace where $x = \bar{x}_1$; likewise $\partial Y(\bar{x}_1, \bar{x}_2, x_3, \ldots, x_n)/\partial x_3$ need be known only for varying $(x_3, \ldots, x_n)$ in the subspace where $(x_1, x_2) \equiv (\bar{x}_1, \bar{x}_2)$; ... finally $\partial Y(\bar{x}_1, \ldots, \bar{x}_{n-1}, x_n)/\partial x_n$ need be known only for varying x_n with the remaining variables fixed. (E.g. in a simple entropy, volume, absolute temperature, pressure system, knowledge everywhere of the pressure-volume-temperature "equations of state" need be supplemented by

knowledge of the specific heat function at one temperature alone to define internal energy and all thermodynamic behavior everywhere.)

Also, a general one-to-one transformation of coordinates, as defined by $x_i = f^i(v_1, \ldots, v_n)$, will lead to new variables $w_1, \ldots, w_n$, conjugate to the v's and determined by the identity $dY = \sum_1^n w_j dv_j = \sum^n y_i dx_i = \sum_1^n [\sum_1^n Y_i(f^1, \ldots, f^n) f_j^i(v_1, \ldots, v_n)] dv_j$.

11. It turns out, however, that we can in principle give an exact test to refute the hypothesis that we have a minimum system. Given a set of finite observations, presumed free of statistical error or with known bounds on that error, we can specify exact inequalities that must be satisfied. Testing an *approximate equality* is in principle a methodologically different problem from the easier case of verifying or refuting an *inequality*. From this viewpoint it is desirable to reverse the usual relative treatment of a minimum and variational principle. In many discussions of classical mechanics, authors are satisfied to state a variational principle without being interested in whether the resulting conditions relate to "stationariness" or to the more difficult question of "definite minimization." From the standpoint of exact empirical refutation, it turns out to be conceptually much easier to test the hypothesis of definite minimization than to test the variational hypothesis. To repeat, this is because an inequality can be empirically refuted more easily than can an approximate equality.

12. Let us consider various distinct equilibrium positions $(x^1; y^1)$, $(x^2; y^2)$, $(x^3; y^3)$, ... Then by definition of the regular minimum

$$Y(x_1^1, x_1^1, \ldots) - \sum_1^n y_j^1 x_j^1 < Y(x_1^2, x_2^2, \ldots) - \sum_1^n y_j^1 x_j^2$$

or

$$Y(x_1^1, x_1^1, \ldots) - Y(x_1^2, x_2^2, \ldots) < \sum_1^m (x_j^1 - x_j^2) y_j^1, \tag{22}$$

showing that R_2' in (19) is necessarily negative. Similarly for R_2''. Hence

$$0 = [Y(x^1) - Y(x^2)] + [Y(x^2) - Y(x^1)] < \sum_1^n (x_j^1 - x_j^2)(y_j^1 - y_j^2); \tag{23}$$

or, letting $x_j^1 - x_j^2 = \Delta x_j$, $y_j^1 - y_j^2 = \Delta y_j$, we have holding between any two distinct equilibrium states

$$\Delta x_1 \Delta y_1 + \Delta x_2 \Delta y_2 + \ldots + \Delta x_n \Delta y_n > 0. \tag{24}$$

In the special case where only one variable of any admissible set of prescribed parameters is allowed to change, we get

$$0 + \ldots + 0 + \Delta x_i \Delta y_i + 0 + \ldots + 0 > 0, \tag{25}$$

showing that conjugate variables must change in the same direction when no change takes place in at least one of every other pair or conjugate variables. (This is a weak form of the so-called Le Chatelier principle.)

If we have m observations, we can apply the above test (24) to each of the $r(r-1)/2$ pairs of observations. If any one of the inequalities fails to be satisfied, then the minimum hypothesis is refuted. However, if we have more than 2 observations, further independent inequalities must be satisfied.

From (22), we have

$$\begin{aligned}\sum_1^n y_j^1(x_j^1 - x_j^2) + \sum_1^n y_j^2(x_j^2 - x_j^3) + \sum_1^n y_j^3(x_j^3 - x_j^1) > 0 \\ = [Y(x^1) - Y(x^2)] + [Y(x^2) - Y(x^3)] + [Y(x^3) - Y(x^1)]\end{aligned} \tag{26}$$

a test that every triad of distinct points must satisfy. Reversing the order of the points, we get a second independent inequality binding the same triad.

Generally, for any circular chain of distinct points $(x^1;\ y^1)$, $(x^2;\ y^2), \ldots, (x^r;\ y^r)$, $(x^{r+1};\ y^{r+1}) \equiv (x^1;\ y^1)$, we have by similar reasoning

$$\sum_{i=1}^r [\sum_j y_j^i(x_j^i - x_j^{i+1})] > 0 = \sum_{i=1}^r [Y(x_i) - Y(x_{i+1})]. \tag{27}$$

Reversing the order of the points from $(1, 2, \ldots, r, 1)$ to $1, r, \ldots, 2, 1$ gives a similar relation

$$0 > \sum_2^{r+1} [\sum_j y_j^i(x_j^i - x_j^{i-1})] = -\sum_1^r [\sum_j y_j^{i+1}(x_j^i - x_j^{i+1})]. \tag{28}$$

By duality, we can infer relations just like (27) and (28), such as

$$\sum_{i=1}^r [\sum_j x_j^i(y_j^i - y_j^{i-1})] < 0; \tag{29}$$

(29) can be derived directly from (28) by rearranging terms, and hence does not represent an independent condition.

The number of independent inequalities is equal to the number of distinct circular chains that can be formed from the finite given number of observations, r. If $r = 2$, we have a single condition, namely (24). If $r = 3$, we can form 3 independent pairings, and hence we have three chains of length two to provide us with inequalities. In addition, we have two independent chains of length three; viz. 123, 321. So altogether for $r = 3$, we have 5 independent inequalities to test.

Given any k points, we can make $N_k = (k-1)!$ circular chains involving all k elements. To prove this, note that $N_2 = 1$, because 12 and 21 are the same circular chain. Also, we go from a chain involving $12 \ldots k-1$ in some order to one involving $12 \ldots, k-1, k$ in some order by putting between *any* two adjacent numbers of the shorter chain the new element k. Since there are k such places to insert the new element, obviously there are k new chains of length k for each chain of length $k-1$. Therefore, $N_k = kN_{k-1}$, a recursion relation yielding, with $N_2 = 1$, the unique solution $N_k = (k-1)!$

The total number of independent inequalities that any r empirically observed points must satisfy is the sum of all possible chain-inequalities of length $k \leqq r$, and therefore equals in number

$$\binom{r}{2}N_2 + \binom{r}{3}N_3 + \ldots + \binom{r}{r}N_r = \binom{r}{2}1 + \binom{r}{3}2 + \binom{r}{4}3! + \ldots$$
$$+ \binom{r}{r-1}(r-2)! + (r-1)! = \sum_2^r \binom{r}{k}(k-1)!$$

For $r = 2, 3, 4, 5$, we already have 1, 5, 20, 74 independent inequalities to be satisfied, and the number rises rapidly with r. (Incidentally, there are the same number of Hicksian equalities, of the type given in (21) that must be approximately satisfied for nearby points.)

This may all be summarized as follows.

Theorem 6: Any r distinct observations $(x^1; y^1), \ldots, (x^r; y^r)$ arising from a definite minimum system must satisfy $\sum_2^r \binom{r}{k}$ $(k-1)!$ independent inequalities of the type

$$\sum_{i=1}^{k} [y_1 \Delta x_1 + y_2 \Delta x_2 + \ldots] > 0, \quad k \leqq r$$

where the summation is over a circular chain consisting of k distinct points and returning to the original point and where the superscripts in y_j^i and in $x_j^i - x_j^{i+1} = \Delta x_j$ have been omitted. Similar, but non-independent, dual relations involving Δy and x can be written.

13. Relations of the type given in Theorem 5 are necessary conditions: violation of any one of them could serve as a refutation of the hypothesis that we are dealing with a regular minimum system. (If we drop regularity assumptions, equalities must be added to the inequalities but otherwise the theorem is still valid.) The question naturally arises as to whether there can be any further necessary conditions beyond the ones already derived. That is, are these necessary conditions in some sense also sufficient conditions for a regular minimum system? The following answer can be given.

Theorem 7: Given n smoothly differentiable relations holding between $2n$ conjugate variables, namely $y_i = Y^i(x_1, \ldots, x_n)$, $i = 1, 2, \ldots, n$, and if there can never arise for any finite number of observations a violation of any of the inequalities of Theorem 6, we are then assured that the system is a regular minimum one, with symmetric positive definite Jacobian matrix and all the other properties of such a system.

To prove this, consider a chain of m distinct points that lie along any simple closed path or contour in the n dimensional $(x_1, \ldots, x_n)$ space, with $1, 2, \ldots, m$ being arrayed by convention in counterclockwise order. Define $S_m = \sum_1^m [\sum_1^n y_j^{i+1}(x_j^i - x_j^{i+1})]$ and $s_m = \sum_1^m [\sum_1^n y^i(x_j^i - x_j^{i+1})]$. By (27) and (28) and our present hypothesis, it is to be true that S_m is positive and s_m is negative for any m. Now let the number of observations, m, along the same simple closed contour increase indefinitely in such a way as to make the distance between any two adjacent points go to zero in the limit. Then clearly from the definition of an integral as the limit of a sum,

$$\lim_{m\to\infty} S_m = \int \sum y_j \, dx_j = \lim_{m\to\infty} s_m$$

where the indicated line integral is taken counterclockwise around the specified simple closed contour and where $y_j = Y^j(x_1, \ldots, x_n)$. Since this line integral is the limit of a sequence of positive numbers and at the same time is the limit of a sequence of negative numbers, its value is proved to be zero along any closed contour. Hence the expression $\sum y_j dx_j$ must be an exact differential $dY(x_1, \ldots, x_n)$, with $y_i = Y^i(x_i, \ldots, x_n) = Y_i(x_1, \ldots, x_n)$ and $Y^i_j = Y^j_i$. This proves the symmetry or variational aspect of the problem.

To prove the definiteness, we may consider the inequality that must hold between any pair of points; given our assumptions concerning differentiability, this may be written in the Taylor's expansion

$$0 < \sum_{1}^{n} \Delta y_j \Delta x_j = \sum_{1}^{n}\sum_{1}^{n} \bar{Y}_{ij} \Delta x_i \Delta x_j$$

where not all Δx_j are zero and where the second partial derivatives are evaluated at some point intermediate between the two points in question. Clearly at such a point the quadratic form made up of the Hessian matrix must be positive definite; and the same must be true of every observed point, else we could write down a contradiction to our finite inequality. This completes the proof.

The interpretation of this theorem is rather delicate. It shows that we can, by taking enough observations close enough together, always succeed in detecting any deviation of the facts from the regular minimum hypothesis. But it does not tell us how many observations will be necessary.[6]) Therefore, no matter how many observations we may have examined, all of which satisfy all requisite inequalities, we cannot be sure but that still further observations might show a violation of the hypothesis. This is a common situation: methodologically, we can

[6]) In *Economica*, February, 1953, p. 9 I posed the "open question" as to how many situations m might be needed to test the existence of an integrable field. Were $n = 2$, we know that $m = 2$ would be sufficient. For $n \geqq 3$ is there a similar finite $m = \Phi(n)$ bound on the number of situations that might be needed to refute integrability? The question can now be closed by the statement: when there is definitely non-integrability, the closer the conditions for integrability are to being fulfilled, the larger must be the m needed to reveal that non-integrability; so there is no *a priori* finite bound on m possible.

refute a hypothesis with a finite number of observations, but we can never strictly "confirm" it in this way. Furthermore, the theorem does not tell us that, for fixed m, satisfying all the prescribed inequalities is all that can be required of the observations: i.e., it does not prove the non-existence of any further independent implications of the regular minimum hypothesis. I would conjecture, from various algebraic and geometric considerations, that no further independent inequalities can be prescribed; but this has not yet been proved.

14. This completes the program of characterization of the necessary and sufficient conditions for a system to have the structure of a regular minimum.[7]) A number of further problems may be briefly mentioned. First, the smoothness and uniqueness assumptions involved in the notion of "regularity" may be dropped, and investigations may be made of the resulting modifications in the theorems. With partial derivatives not being defined at sets of points, the Jacobian and Hessians of the first part of this paper will then not necessarily exist or be

[7]) The rather ambiguous principle of Le Chatelier, in its correct form

$$x_1^1(y_1, \ldots, y_n) \geqq x_1^1(y_1, \ldots, y_{n-1}; x_n) \geqq \ldots \geqq x_1^1(y_1; x_2, \ldots, x_n) \geqq 0$$

can be derived from our conditions, as can the equivalent dual form which says that $y_n^n(y_1, \ldots, y_m; \ldots, x_n)$ must be positive and non-decreasing with m. But adding the same finite positive Δy_1 to each $(y_1, \ldots, y_m; x_{m+1}, \ldots, x_n)$, all of which correspond to the same initial equilibrium point, *need not* lead to the finite inequalities

$$x^1(y_1+\Delta y_1, \ldots, y_n) \geqq x^1(y_1+\Delta y_1, \ldots, y_{n-1}; x_n) \geqq \ldots \geqq x^1(y_1+\Delta y_1; x_2, \ldots, x_n).$$

For sufficiently *small* changes in y_1 these finite inequalities must hold, but the proof is not immediate. For finite moves of any size, it is easy to give counter-examples that are nonetheless perfectly regular. This may be regarded as slightly paradoxical since it says that from any point in the (pressure, volume) diagram, there will be an isothermal line "flatter" than an adiabatic line; yet, the finite change in pressure in a system when you change its volume holding temperature constant, can turn out to be greater than the change in pressure resulting from an equal change in volume with entropy being held constant. The paradox is disspelled if one realizes that

$$\int_0^{\Delta y_1} \{x_1^1(y_1 + t, \ldots, y_m; \ldots, x_n) - x_1^1(y_1 + t, \ldots, y_{m+1}; \ldots, x_n)\}dt$$

has an integral that starts out non-negative at $t = 0$ but which may become negative as the equilibrium points being compared become different ones. The last vestige of paradox disappears when we realize that regular systems need not admit of (p, v) as prescribable coordinates; hence, any point in this

invertable. Extensions of the Implicit Function Theorem and of the concept of one-to-one correspondence will lead to results, of which the regular systems constitute one special case. As far as the finite inequalities of the latter part of the paper are concerned, with the addition of equalities to the inequalities, they will all remain valid.

A second, quite different extension of the present program is to apply it to the study of regular systems involving an infinite number of variables or the equivalent. Thus, the differential equations of classical mechanics can be regarded as functional systems that are limiting cases involving an infinite number of variables. The question naturally arises as to an exhaustive summary of the necessary and sufficient conditions that the observable motions must satisfy if they are to have arisen from a regular variational system that is or is not a true minimum system. As far as I know, no one has yet carried through such a program in the full detail of the present examination of the simpler case of a finite number of functions of a finite number of variables. The present discussion shows some of the problems that will certainly be encountered in the more difficult realm of function spaces.

diagram may correspond to more than one physical state of the system.

A slight generalization of the Le Chatelier principle can be stated in terms of quadratic forms, namely

$$\sum_{1}^{r}\sum_{1}^{r} x_j^i(y_1, \ldots, y_m; \ldots, x_n) h_i h_j \geqq \sum_{1}^{r}\sum_{1}^{r} x_j^i(y_1, \ldots, y_k; \ldots x_n) h_i h_j,$$

where $r \leqq k < m$. This is provable directly from the form of J_n in (6) and its dual.

APPENDIX

Axiomatic Basis For Equilibrium Relations in Classical Thermodynamics.

1. The formal mathematical analogy between classical thermodynamics and mathematic economic systems has now been explored. This does not warrant the commonly met attempt to find more exact analogies of physical magnitudes — such as entropy or energy — in the economic realm. Why should there be laws like the first or second laws of thermodynamics holding in the economic realm? Why should "utility" be literally identified with entropy, energy, or anything else? Why should a failure to make such a successful identification lead anyone to overlook or deny the mathematical isomorphism that does exist between minimum systems that arise in different disciplines?

In this Appendix, some of the special differences between the variables and relations that arise in thermodynamics and in economics will be briefly explored. This will involve giving a slightly unconventional axiomatic base for classical thermodynamics, or at least for that part which deals exclusively with equilibrium states and which might be called thermostatics.[1])

2. While in economics the variables (x_i, y_i) will usually be observable quantities of goods and their respective prices, in thermodynamics they will not all be as immediately observable as is the case of volume and (negative) pressure. Thus, (x_1, y_1) might correspond to "entropy" and "absolute temperature," which to early writers were not known, directly observable quantities. And a function like $Y(x_1, \ldots, x_n)$, which in economics might be easily measured as dollar revenue, in thermodynamics

[1]) Such an axiomatic excursion departs from the usual Clausius formulation, in terms of the first and second laws. (Cf. also the work of Carnot, Kelvin, Gibbs, and Planck). It resembles a little the alternative axiomatic approach of C. Carathéodory, *Mathematische Annalen*, 67 (1909) pp. 355—86. See M. Born, *Natural Philosophy of Cause and Chance* (Oxford, 1949), Ch. V for a good account and for references.

might correspond to an internal energy function whose existence and properties have to be established by intricate reasoning.

Fig. 1 plots the properties of a simple system whose states can, for simplicity, be assumed to be determined by pressure and volume (p, v). The solid contours represent loci of equal "temperature" and as yet have no natural, preferred numbering on them corresponding to absolute temperature, t. Instead $T = T(v, p)$ represents any monotone numbering; and any renumbering $T' = f(T) = T'(v, p)$, with f an arbitrary monotone-increasing function, would be an equally good indicator of greater and smaller temperatures.

The broken contours represent "adiabatics" and represent loci of points that could be observed if the system were "insulated" from its environment by perfect non-conducting walls. I omit a good deal of explanatory matter needed to establish the exact nature of these two sets of contours, instead taking them as the given primitive concepts of the axiomatic system. Note that it would be premature to call the adiabatics "isentropes" since we do not yet have any entropy magnitude to be held constant. Yet we can give an arbitrary numbering to the adiabatics, written as $S = S(v, p)$ or as $S' = F(S) = S'(v, p)$, where again F is an arbitrary monotone-increasing function which will preserve the direction of the ordering.

3. The first task of our thermodynamic system is to define its own "canonical entropy and temperature" so that we can go from the arbitrary scaling of T and S to a privileged canonical temperature t and canonical entropy s; when this is done we shall be able to define an internal energy function $e = e(v, p)$ or $e = E(s, v)$.

Note that Fig. 1 is drawn with the special property of "proportional areas." Thus the shaded area A is to B as the area A' is to B'. And the same equal proportionality would be true of the appropriate curved parallelograms formed by any four thermal contours intersecting with any four adiabatics.

This equal proportionality of areas is not an accident: it is a fundamental regularity of nature, from which we can deduce the existence of an internal energy, a canonical temperature and canonical entropy.

4. In fact pick any thermal contour *aa* as the arbitrary origin for temperature and any other thermal contour *bb* as the arbitrary $t = 1$ level which will set the scale of one unit of canonical

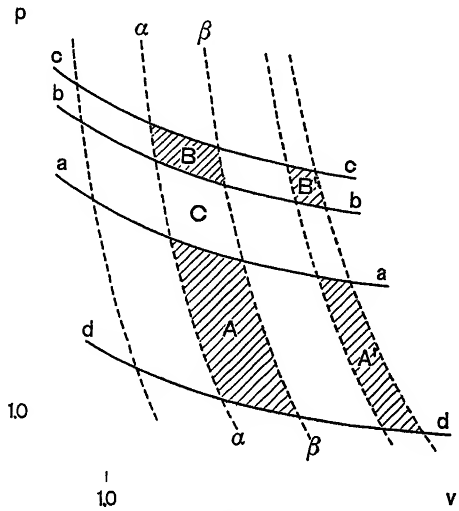

Fig. 1

temperature. We can now give a determinate number t to any arbitrarily given thermal contour by making its t the ratio of the algebraic area that it and any pair of adiabatics form with *aa* relative to the area formed by the unit temperature contour. Fig. 2. shows the resulting canonical temperature scaling for each contour. (Verify that *cc* corresponds to $t = 1.5$ because the area $C + B$ is one and a half times as great as the area C alone; of course $(C' + B')/B'$ gives the same ratio. And *dd* corresponds to $t = -2.0$ because the area A *below aa* is twice the absolute size of the area C.)

5. Having defined canonical t, we have no further use for T or T'. And we can now go from S or S' to canonical s, called entropy. This may be measured from *any* point selected as an arbitrary origin: thus the point northwest of A may be taken as zero origin, although it is to be understood that this entropy origin would not have to be at a point where $t = 0$.

Once the dimension of pressure is selected, the scale of s is

not arbitrary. Suppose $\beta\beta$ has been selected just the proper distance from $\alpha\alpha$ as to make the area C (which has the dimension of pressure times volume) be exactly unity in area. Then $\beta\beta$ is given the designation $s = 1.0$ and every other adiabatic is given a unique s determined by the algebraic value of the area which it makes with aa, bb, and $\alpha\alpha$. (If the adiabatic is to the left of $\alpha\alpha$ we give it a negative entropy with magnitude proportional to appropriate absolute area. See Fig. 2 for illustrative s values.)

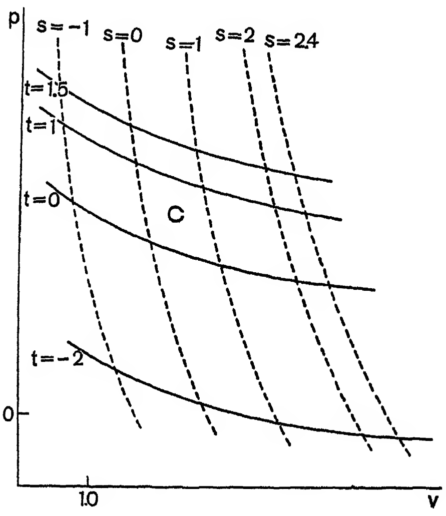

Fig. 2

6. Now that we have defined the functions $t = t(v, p)$ and $s = s(v, p)$, we can verify that the equal proportionality of area property implies that the line integral

(A1) $$t(v, p)\, ds(v, p) - p dv$$

must be independent of path and expressible as the exact differential $de(v, p)$ of an existent "canonical internal energy" function $e = e(v, p)$, which is arbitrary in its origin because of the constant of integration.

Because the canonical temperature t is arbitrary in its origin, transforming from t to $t + t_0$, would introduce an arbitrary linear entropy term $t_0(s - s_0)$ in our expression for canonical energy.

Within the realm of thermostatics, where we merely observe equilibrium states and make no measurements of "irreversible processes," we can never hope to remove this ambiguity in the definition of canonical internal energy.[2] We can never hope to deduce the level of absolute zero, which characterizes the Kelvin scale. We cannot define the "efficiency" of an hypothetical Carnot cycle from knowledge of the isotherms and adiabatics alone. (In fact the present treatment never explicitly mentions "heat," inexact differentials like dQ, or exact differentials like dQ/t.)

7. Under our simplifying assumption that (v, p) are admissible state variables, we can rewrite internal energy e as a function of the pair of variables (s, v) to get $E(s, v)$ such that

$$dE(s, v) = t(s, v)ds - p(s, v)dv. \tag{A2}$$

Here $E(s, v)$ corresponds precisely to my earlier $Y(x_1, x_2, \ldots)$; and it would be easy to show that the Gibbs potential $e - ts - (-p)v$, written as a function of t and $-p$, corresponds to the dual $X(y_1, y_2, \ldots)$. Corresponding to $P(y_1; x_2, \ldots)$ would be the Helmholtz free energy $e - ts$ written as a function of t and v.

8. All of the above implications of the equal-proportional areas axiom can be given brief mathematical summary[3] in the case where all the functions have nice (overly strong) differentiability and regularity properties: in particular where any two $T(s, p)$ and $S(v, p)$ contours intersect in one and only one point so that these functions can be inverted to give $v = v(S, T)$ and $p = p(S, T)$ with smooth partial derivatives.

Suppose we know that indicators of temperature and entropy can be converted by determinate (save for arbitrary origin constants and scale constants) functions $t = f(T)$ and $s = F(S)$ to yield canonical temperature and energy; and that there exists a canonical internal energy function $e = E(s, v)$ with the Maxwell reciprocity or integrability property

[2]) This was noted by Carathéodory, *op. cit.*, p. 381. See also A. Landé, *Handbuch der Physik*, Band IX, 281—300 for a treatment related to the present one.

[3]) This and the next section can easily be skipped.

$$\text{(A3)} \qquad \frac{\partial^2 E(s, v)}{\partial v \partial s} \equiv \frac{\partial^2 E(s, v)}{\partial s \partial v}.$$

From the form of the exact differential in (A2), this is equivalent to

$$\text{(A4)} \qquad \frac{\partial t(s, v)}{\partial v} \equiv -\frac{\partial p(x, v)}{\partial s};$$

which in terms of arbitrary T and S and $f(T)$ and $F(S)$ becomes

$$\text{(A5)} \qquad \frac{\left(\dfrac{\partial p}{\partial S}\right)_v}{\left(\dfrac{\partial T}{\partial v}\right)_S} \equiv f'(T)\, F'(S).$$

Now the left-hand side of this relation is a perfectly observable magnitude, being a function of (v, p) or (S, T) or any other two variables. Let us call it J. The equal-proportional area property — which is an integrability property in the large — is equivalent to saying that this observable left-hand expression J is truly a *product* of two separate functions. I.e., necessarily $\log J$ must be the sum of a function of T and a function of S, so that

$$\text{(A6)} \qquad \frac{\partial^2 \log J(S, T)}{\partial S \partial T} \equiv 0$$

is a necessary and sufficient condition for the equal-proportional area property to hold.

An alternative functional equation which J must satisfy is

$$\text{(A7)} \qquad \begin{vmatrix} J(S_1, T_1) & J(S_2, T_1) \\ J(S_1, T_2) & J(S_2, T_2) \end{vmatrix} \equiv 0$$

for all (S_1, T_1) and (S_2, T_2).

Both (A6) and (A7) are observable conditions which could be empirically verified.

Incidentally, by some manipulations of implicit function theory, it is easy to show that J is a certain Jacobian determinant, namely

$$\text{(A8)} \qquad \frac{\left(\frac{\partial p}{\partial S}\right)_v}{\left(\frac{\partial T}{\partial v}\right)_S} \equiv \frac{1}{\begin{vmatrix} \frac{\partial S}{\partial v} & \frac{\partial S}{\partial p} \\ \frac{\partial T}{\partial v} & \frac{\partial T}{\partial p} \end{vmatrix}} = \begin{vmatrix} \frac{\partial v}{\partial S} & \frac{\partial v}{\partial T} \\ \frac{\partial p}{\partial S} & \frac{\partial p}{\partial T} \end{vmatrix} \equiv \frac{\partial(v, p)}{\partial(S, T)} \equiv J(S, T).$$

9. Given the observable function $J(S, T)$, we can derive canonical temperature and entropy by simple integrations. Thus

$$\text{(A9)} \qquad \begin{aligned} t = f(T) &= t_0 + \int_{T_0}^{T} f'(\tau)d\tau \\ &= t_0 + f'(T_0)\int_{T_0}^{T} \frac{J(S_0, \tau)}{J(S_0, T_0)}\,d\tau, \end{aligned}$$

where $t_0 = f(T_0)$ is an arbitrary origin constant and $f'(T_0)$ is an arbitrary scale constant. Also

$$\text{(A10)} \qquad \begin{aligned} s = F(S) &= s_0 + \int_{S_0}^{S} F'(\sigma)d\sigma \\ &= s_0 + \int_{S_0}^{S} \frac{J(\sigma, T_0)}{f'(T_0)}\,d\sigma \end{aligned}$$

Because J has the remarkable multiplicative property, this $f(T)$ integral will be independent of the S_0 level and (A10)'s $F(S)$ integral will necessarily be independent of the T_0 level. Note that the origin of s is arbitrary, but the scale is definitely not.

Since the line integral $tds - pdv$ is independent of path, we can by a variety of alternative integrations calculate the canonical internal energy function $E(s, v)$, which is determinate except for an arbitrary linear term $t_0(s - s_0)$.

10. So far the discussion dealt with simple systems with only two degrees of freedom: e.g. a single homogeneous fluid. The existence of an absolute or Kelvin *temperature which is the same for all substances* shows that the equal-proportional area law must apply also to the interrelations of different bodies. Thus Fig. 3 shows that if two bodies are in respective equilibrium at $t = 0$ and at $t = 1$, then we can separately for each calculate the canonical temperatures $t_1 = \frac{1}{2}$ and $t_2 = \frac{1}{2}$ by our previously

described techniques of canonical temperatures. Then we can confidently predict — in advance of ever having made the experiment! — that the $t_1 = \frac{1}{2}$ and $t_2 = \frac{1}{2}$ contours will truly represent the same temperature and represent mutual equilibrium.

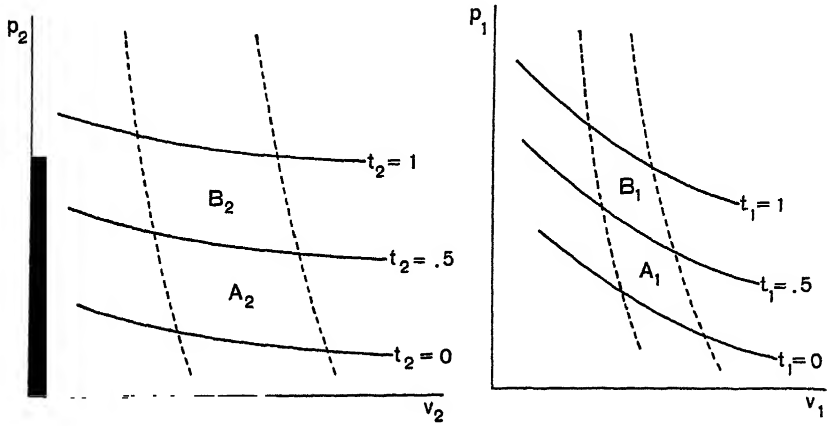

Fig. 3

This requires that the area relations $A_1/B_1 = A_2/B_2$ hold between different substances. (In fact this is the generalized form of our equal-proportional area axiom, from which all the other relations of thermostatics follow.)

11. There are far-reaching implications of the axiom. Thus, suppose two (or more) bodies are separated by *diathermous* walls (i.e. walls which require the bodies' mutual equilibrium to be at equal temperatures) and that these walls are also *elastic* (i.e. such as to require equal pressures at equilibrium). Then

$$\text{(A11)} \qquad \begin{aligned} t_1(s_1, v_1) &= t_2(s_2, v_2) = \ldots = t = f(T) \\ p_1(s_1, v_1) &= p_2(s_2, v_2) = \ldots = p. \end{aligned}$$

But in terms of the internal energies $E_i(s_i, v_i)$, this means

$$\text{(A12)} \qquad \begin{aligned} \frac{\partial E_1(s_1, v_1)}{\partial s_1} &= \frac{\partial E_2(s_2, v_2)}{\partial s_2} = \ldots = t \\ \frac{\partial E_1(s_1, v_1)}{\partial v_1} &= \frac{\partial E_2(s_2, v_2)}{\partial v_2} = \ldots = -p, \end{aligned}$$

which in turn is seen to be the necessary condition that $e = e_1 + e_2 + \ldots = E_1(s_1, v_1) + E_2(s_2, v_2) + \ldots$ be at a minimum subject to

$$\text{(A13)} \qquad \begin{aligned} s_1 + s_2 + \ldots &= s \\ v_1 + v_2 + \ldots &= v; \end{aligned}$$

and that the resulting minimized energy be a function of the total s and v respectively — namely

minimum $e = E(s_1 + s_2 + \ldots, v_1 + v_2 + \ldots) = E(s, v)$ with

$$\text{(A14)} \qquad \frac{\partial E(S, v)}{\partial s} = t, \quad \frac{\partial E(s, v)}{\partial v} = -p,$$

where again an arbitrary linear term in total entropy $t_0(s - s_0)$ will be involved.[4])

For a true maximum, it is sufficient that each $E_i(v_i, s_i)$ have a positive definite Hessian matrix of second partial derivatives

$$H_i = \begin{bmatrix} \dfrac{\partial^2 E_i}{\partial s_i^2} & \dfrac{\partial^2 E_i}{\partial v_i \partial s_i} \\ \dfrac{\partial^2 E_i}{\partial s_i \partial v_i} & \dfrac{\partial^2 E_i}{\partial v_i^2} \end{bmatrix},$$

and then it will follow that the Hessian matrix of $E(s, v)$ is positive definite.[4a])

[4]) If the bodies are in thermal contact but separated by *rigid* walls, the pressure equality of (A12) is lost and the volumes $(v_1, v_2, \ldots)$ are separately specifiable: we then get $e = e_1 + e_2 + \ldots = E(s_1 + s_2 + \ldots, v_1, v_2 \ldots)$ in minimized form, with $\partial E/\partial v_i = -p_i$ and $\partial E/\partial s = t$.

[4a]) Many treatises on thermodynamics seem to be saying that $E(s, v)$ is at a minimum for prescribed s and v. Actually $E(s, v)$ is a determinate function of its arguments and is not free to be at a maximum or a minimum. It is the sum $E_1(s_1, v_1) + E_2(s_2, v_2)$ which has its (s_i, v_i) varied subject to fixed $s_1 + s_2$ and $v_1 + v_2$ until $E_1 + E_2$ is at a minimum. When the resulting optimal (s_i, v_i) are substituted into $E_1(s_1, v_1) + E_2(s_2, v_2)$ the result defines $E(s, v)$.

That the Gibbs free energy or potential is at a minimum for fixed t and p likewise requires careful statement. Write the expression $e - ts - (-p)v$, regarded as a function of t and p alone, as $G(t, p)$. Then $G(t, p)$ cannot be at a minimum for arbitrarily prescribed (t, p), nor at a maximum, nor anything else but at its determinate value. Rather it is $G(s, v; t, p) = E(s, v) - ts - (-p)v$ that is at a minimum with respect to (s, v) for prescribed (t, p). This can be shown to be a special case of the condition that $E_1 + E_2$ is at a minimum for prescribed $v_1 + v_2$ and $s_1 + s_2$. Think of s, v and E as belonging to one body, and hence write them as s_1, v_1, $E_1(s_1, v_1)$. Now consider a second body so large compared to the first that its temperature and pressure will be affected negligibly by

12. Note that in contrast to the Carathéodory-Born treatment, there was here no stipulation of any integrability conditions or inaccessibility conditions other than that of equal-proportional area. Once the $T_i = T_i(v_i, p_i)$ and $S_i = S_i(v_i, p_i)$ contours were specified and the equal-proportional area axiom of a single uniform canonical temperature was specified, then all the remaining relations of thermostatics logically followed.

13. Instead of minimizing total energy e for given total entropy s and total volume, with $t > 0$ we could have maximized entropy for given total energy and volume. Leaving thermostatics for true thermodynamics and actually observing the time changes of a given *isolated* system, we would actually find that total energy would remain constant and total entropy would increase in time. But within the realm of thermostatics — i.e. the study of equilibrium relations like (A11) — we would have no knowledge of this or interest in it.

14. The present formulation [5]) thus, seems to have turned the usual formulation upside down. Rather than begin with the first and second laws of thermodynamics and then deduce equilibrium relations and the existence of a universal absolute temperature, we have assumed a universal canonical temperature (with arbitrary origin) and have then deduced equilibrium relations and integrability conditions. We have assumed less than the first and second laws: and we deduce less. We deduce only thermostatic relations, deducing almost nothing about irreversible processes. (In the original definition of the adiabatics $S_i = S_i(v_i, p_i)$ there does lurk in the background an

the first. (A "heat and pressure bath.") Write its energy as $E_2(s_2, v_2) \equiv \bar{t}s_2 - (-\bar{p})v_2$, where $\bar{t}$ and $\bar{p}$ are fixed parameters. Now minimize $E_1(s_1, v_1) + E_2(s_2, v_2)$ subject to fixed $\bar{s} = s_1 + s_2$ and $\bar{v} = v_1 + v_2$ to get

$$\text{Minimum } E_1(s_1, v_1) + E_2(s_2, v_2) = [E_1(s_1, v_1) - \bar{t}s_1 - (-\bar{p})v_1] + \bar{t}\bar{s} + (-\bar{p})\bar{v}$$

$$= G_1(s_1, v_1; \bar{t}, \bar{p}) + \text{a constant.}$$

This last expression shows that minimizing the Gibbs G_1 is a special case of minimizing $E_1 + E_2$, the total of the observed system *and* its "environment."

[5]) The equal-proportional area property on which all this is based was suggested to me by an obvious extension of the reasoning of Maxwell's *Theory of Heat* (1871 edition), but I have not seen it explicitly alluded to in any twentieth century book. It is, however, so obvious and basic a property that it must have been rediscovered many times.

implicit reference to irreversible processes and *two-dimensional* "inaccessible neighboring states." But once the primitive notion of an adiabatic is assumed, all the rest is thermostatics.)

The Gibbs thermodynamics of heterogeneous substances has not been here discussed. But what was done here for $-\int p\,dv$ areas could also be done for $-\int \mu_i dM_i$ areas, where μ_i represents chemical potentials and M_i chemical masses; and it would seem that no new mathematical problems would be raised.

KENNETH J. ARROW and LEONID HURWICZ *

Decentralization and Computation in Resource Allocation

I. FORMULATION OF THE PROBLEM

A. *Introduction.*

In this paper, we wish to discuss the bearing of some recent developments in mathematical economics on the problem of the optimal allocation of resources. We will confine attention here to an economy whose aims are well defined. That is, we assume that the preferences of the economic system can be embodied in a utility function which depends upon the outputs of commodities. For a given technology, the possibilities of different output combinations are restricted by the availabilities of primary resources. The problem of optimal resource allocation is to choose among all the feasible combinations of production processes that combination which maximizes the utility achieved by the economy.

Since the discussion is at a fairly high level of abstraction, the economy being studied may be a nation or some smaller economic system, including a single firm. The assumption that a single utility function represents the objectives of the economy fits best the case of a firm. For a nation, the assumption is less justified, but it provides an introduction, at least, to the more complex problem raised by the presence of many individuals, each of whom judges the workings of the economic system in light of his own utility function. We also avoid the subtle problems involved in defining optimality in the more general case.

The problem of choosing the allocation of primary resources

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among different productive processes so as to maximize a prescribed utility function is a mathematical one, and its solution in any concrete case can be regarded as a matter of computation. Now any computation, beyond the very simplest, involves a process of successive approximations, that is, a process carried out in several steps, where the calculations in each step make use of the results of the preceding steps and one of the results of each step is an approximation to the desired answer. (The ordinary process of long division is an example of a process of successive approximations; the results of each trial division step are another digit to be added to the quotient and a new dividend to be used in the next trial division.)

That the market place solves the economic problem of equating supply and demand by successive approximations to the equilibrating price or prices is a concept familiar from elementary economics textbooks. In a single market, each approximation results in naming a price and calculating the difference between demand and supply at that price; the next approximation involves adjusting the previous trial price in a manner governed by this difference, with the idea of ultimately wiping it out. The notion of this kind of dynamics in the movement of prices has its roots in the English classical economists, perhaps particularly John Stuart Mill, but it received its first explicit recognition, particularly with reference to simultaneous successive approximations to all prices, in the concept of *tatônnements* of Léon Walras.[1])

Since welfare economics assures us that under certain assumptions as to the utility function and the productive process (see part III, section A below for a more explicit discussion of these assumptions) a competitive equilibrium can be identified with an economic optimum, we may conclude that the method of successive approximations which solves the problem of market

[1]) See L. Walras, *Elements of Pure Economics*, ed. and tr. by W. Jaffé, (London: George Allen and Unwin, 1954), pp. 84—86, 90—91, 105—6, 169—172, 243—54, and 184—95. For a vigorous and stimulating analysis of Walras's theory of *tatônnements* and its central function in seeking to show that economic equilibrium is solved in the market by successive approximations, see D. Patinkin, *Money, Interest, and Prices* (Evanston, Illinois, and White Plains, New York: Row, Peterson and Company, 1956), pp. 377—85.

equilibrium is also a computational method for solving the problem of optimal resource allocation. Indeed, it was seen in precisely this light by Pareto[2]), who compared the market to a computing machine. Of course, as soon as the problem of optimal resource allocation is formulated as the solution of a system of simultaneous equations (namely, the equations defining competitive equilibrium), the possibility arises of solving them by some centralized procedure involving the use of computing machines rather than the market. Pareto objected that the enormous number of equations made such a procedure impossible. A completely centralized organization would require a capacity for storage and processing of technological and other information that exceeds anything likely to be available. The competitive process, on the contrary, achieves *decentralization.*[3]) At each stage in the market's process of successive approximations, any individual firm adjusts its tentative production plans making use of information only about the current tentative prices and its own technology. The adjustments of tentative prices, at the same time, depend only on the aggregate demands and supplies. These are simply a sum of the tentative production plans of the individual firms (and consumption plans of consumers) plus the originally existing supplies of basic resources.

Thus the information needed by firms and consumers consists solely of their technologies or utility functions plus prices, while the adjustment of prices is based only on the aggregate of individuals' decisions. It is the minimization of information requirements for each participant in the economy which constitutes the virtue of decentralization.

Our aim here is to state more precisely than hitherto the

[2]) V. Pareto, *Manuel d'Économie Politique* (Deuxième Édition, Paris: Marcel Giard 1927), pp. 233—34. In the terminology of cybernetics, Pareto, crystallizing the usual views of economists, has held that the market mechanism is homeostatic, a position strongly denied by the founder of cybernetics; see N. Wiener, *Cybernetics* (New York and Paris: John Wiley and Sons, and Hermann et Cie., 1948), pp. 185—86. The comparison between the market and a computing machine has also been made by R. M. Goodwin, "Iteration, Automatic Computers, and Economic Dynamics," *Metroeconomica*, III (1951), 1—7.

[3]) For a possible definition of the term "decentralization" in this context, see L. Hurwicz, "Decentralized Resource Allocation," Cowles Commission Discussion Paper No. 2112, 1955. See also the first paragraph of p. 77 below.

dynamic system which is implied by the market mechanism, to give conditions under which the resource allocation determined by it converges to the optimal, where optimality is defined in terms of a single utility function for the economy, and to suggest modifications of the market mechanism which still preserve some degree of decentralization for cases where the conditions in question are not satisfied. The conditions for convergence of the unmodified market mechanism are basically those of diminishing or constant returns in production and diminishing marginal utility (in a generalized sense) for the consumption of final demands. To study these dynamic problems, we will make use of a variety of mathematical tools, some of which have arisen in the theory of games and of linear and nonlinear programming and some of which are more classical applications of differential equations to maximization problems.

In section B of this part, we will review the history of the problem of dynamic adjustment to a social optimum from its initial formulation by Walras. In section C, we will state more precisely our resource allocation model. In part II, the mathematical tools that will be used are reviewed; they are primarily the gradient method (or method of steepest ascent), which is a system of differential equations for solving a maximization problem, and the relation between constrained maxima and saddle points. In part III, we study the case of production under diminishing or constant returns with a strictly concave utility function (see Definition 1 below; the condition is a generalized form of diminishing marginal utility). In this case two dynamic systems which formalize the intuitive notion of a market mechanism have been shown to converge. In part IV, we consider the cases where unmodified market processes do not converge, in particular, that of increasing returns. Here three modified market mechanisms are proposed and shown to have some desirable properties. One such mechanism is closely related to imperfect competition, another to speculation. It is also shown that the purely linear case (linear utility function and constant returns to scale), for which the unmodified market mechanisms do not converge, can be completely solved by some of the modified mechanisms.

B. *Historical Remarks.*

Most of the discussions in the economic literature on the achievement of a social optimum through the market, whether in a socialist or a capitalist economy, have contented themselves with a static characterization.[4] The main contributors to a dynamic formulation of the market mechanism have been Walras, Pareto, Taylor[5], Lange,[6] and Samuelson.[7]

In the present study, we are concerned with a single consumer in whose interests the whole economy is run. Walras, Pareto, Taylor, and Samuelson are all concerned with a multiplicity of consumers; Lange deals with both cases. In the many-consumer case, it is assumed that the consumer maximizes his utility instantaneously, so that his demand for commodities is a given function of prices.

Walras was not explicitly concerned with a socialist economy, but he did regard the competitive system as a computing device for achieving a maximum of satisfaction to society. On the production side, he assumed that all production processes were linear-homogeneous.[8] In this case, the profit is proportional to the scale, at any given set of prices, and the marginal profitability is simply a constant. Walras' rule for adjustment when the

[4]) See for example A. P. Lerner, *The Economics of Control* (New York: Macmillan, 1946) or J. E. Meade, *Planning and the Price Mechanism* (London: George Allen and Unwin, 1948).

[5]) F. M. Taylor, "The Guidance of Production in a Socialist State," *American Economic Review*, Vol. 19 (1929), No. 1, reprinted in B. Lippincott (ed.), *On the Economic Theory of Socialism* (Minneapolis: University of Minnesota Press, 1938), pp. 41—54, particularly pp. 50—54.

[6]) O. Lange, "On the Economic Theory of Socialism," *Review of Economic Studies*, IV, Nos. 1 and 2 (1936-7) reprinted in B. Lippincott, *ibid.*, pp. 57—142, particularly pp. 70—98.

[7]) P. A. Samuelson, *Foundations of Economic Analysis* (Cambridge, Mass.: Harvard University Press, 1947) pp. 269—75.

[8]) We follow the discussion in Walras, *op. cit.*, Lesson 22, which treats the adjustment process when there are both consumers and producers. In this lesson, it is also assumed that there is only one output for each process and only one process for producing each commodity (i.e., the assumption of fixed production coefficients), but this assumption is irrelevant to our purposes. Later, Walras considers the production coefficients not as given but as derived by minimization of costs with a given production function (pp. 383—86); but he never successfully integrates this discussion with the earlier discussion of adjustment, as Jaffé points out (pp. 552—53).

economic system is in an initial state of disequilibrium is that (a) the price change has the same sign as the excess demand, and (b) the change in the scale of each process has the same sign as the marginal profitability.[9])

Walras' adjustment rule for firms is an inevitable consequence of his assumption of constant returns to scale. We cannot postulate that a firm will instantaneously maximize its profits at any given level of prices; under constant returns, the profit-maximizing scale may be infinite, if positive profits are possible at some level, or it may be that there are zero profits at all scales, in which case profit-maximization does not define the behavior of the firm, or, if profits are negative at all positive scales, the optimal scale is zero. Walras correctly saw that under these circumstances he could not prescribe instantaneous-profit-maximization in the way he did require instantaneous utility-maximization by the consumer. Despite the necessary character of lagged adjustment, Walras' successors were usually not so careful, and a clearer understanding of the special adjustment problems connected with constant returns to scales was not achieved until the development of game theory and linear programming brought them to the fore.

Pareto explicitly noted that a socialist system would have to mimic the process of competition to achieve an optimal allo-

[9]) Walras, *op. cit.*, pp. 253—54; the terminology has been changed to conform to that used below. The reader should be warned that Walras' presentation is by no means unequivocal, and the interpretation is not the only one possible. We have followed the statement which summarizes the lesson (22) on *tatônnements* in an economy with consumption and production. But the preceding ten pages, if taken literally, present an adjustment process only distantly related to the summary. Some prices are held constant while others vary independently to clear different markets; similarly quantities are adjusted to make profits zero in each industry, but the adjustment process described takes all prices other than the selling price as given. It is hard to believe that Walras meant to describe reality as if markets and firms came into equilibrium in a preassigned order. It is perhaps this difficulty which led Goodwin (*op. cit.*, p. 5) to deny that Walras meant his adjustment process to be practical, as describing either reality or a device for the operation of a socialist society. However we feel that Walras suffered in his exposition from the crudity of his mathematical tools; he was seeking to explain a simultaneous adjustment in many markets and within many firms without using the concept of a system of differential (or difference) equations and hence was forced to resort to a crude formulation in which some variables changed while others are held constant.

cation[10]) and, following Walras, laid great emphasis on the market as a computing device to solve the system of equations of general economic equilibrium. However, his description of the dynamics of adjustment is considerably less precise than Walras'. In the *Cours* (published in 1896) he indicated rather sketchily his agreement with Walras' description of the competitive process[11]) and, while agreeing with Edgeworth that computationally there are many ways of achieving an economic optimum, contends that Walras' way is the natural economic way.[12])

The description in the later *Manuel* is more extensive but more obscure. The discussion of stability implies a dynamic system in which price responds to a difference of supply and demand.[13]) However, the meaning of the supply functions for firms is never clearly defined. For those operating under diminishing returns, it is the ordinary profit-maximization rule; for others, it appears rather to be the rule that price equals average cost.[14]) The justification for the latter however is based on free entry, which is itself a dynamic process akin to Walras', and should have been introduced explicitly.

Taylor distinguished between primary factors and produced commodities. For any given set of prices of the former, the price of the latter are set so that price equals cost of production.[15]) The demand by consumers at these prices indirectly generates a demand for primary factors. The prices of these are then increased if demand exceeds supply, decreased in the opposite case.[16])

Taylor's rule for price-setting by producers is anything but clear. In the special case where constant returns to scale prevail, there is no joint production, and each firm produces a final product directly from primary factors, Taylor's rule is unequi-

[10]) *Manuel, op. cit.*, pp. 362—64. There is an amusing misprint, where he speaks of a collectivist society "qui ait pour but de procurer à ses membres le *minimum* d'ophélimité" (p. 362, italics added).

[11]) *Cours d'économie politique* (Tome Premier, Lausanne, Paris, and Leipzig: F. Rouge, Pichon, and Duncker and Humblot, 1896), pp. 45—47.

[12]) *Ibid.*, p. 25.

[13]) *Manuel*, pp. 223—24, 232—33.

[14]) *Ibid.*, pp. 177—79, 185—87.

[15]) Taylor, *op. cit.*, p. 45.

[16]) Taylor, *op. cit.*, p. 53.

vocal. However, in the absence of constant returns, average costs will depend upon a system of simultaneous equations involving the demand functions among others. Even with constant returns, the setting of prices by Taylor's rule require solution of a system of simultaneous equations if products of some firms are used by others. If there are alternative processes for producing the same commodity, there would also be a minimization problem to solve.[17]) No indication is given as to the solution of these problems; certainly there would have to be some degree of centralization, in the sense that a central productive agency would have to have access to all the technical coefficients, at least, in order to determine prices.

Lange has been the strongest defender of the proposition that a socialist economy can achieve both an optimal allocation of resources and the computational and informational virtues of decentralization. He presents two adjustment models, according as there is or is not consumers' sovereignty. In both cases, the behavior of firms is defined by the principle of marginal-cost pricing. That is, at given prices, the firm is supposed first to find for any given output the minimum cost of producing it. The output is then determined so as to equate marginal cost, so calculated, to price. The demand functions for factors are determined by the cost-minimization criterion.

This formulation is clearly designed to encompass the cases of both increasing and decreasing returns. In the latter case, it corresponds in general to the single rule of choosing inputs and outputs so as to maximize profits. As we shall see below,[18] the rule is inadequate and strictly speaking incorrect for the case of increasing returns. Further, the rule clearly does not meet the problems raised by the case of constant returns, as sketched above.

[17]) If there is no joint production, then the efficient choice of processes will be the same for all demand functions. See T. C. Koopmans, ed., *Activity Analysis of Production and Allocation*, Cowles Commission Monograph No. 13 (New York and London: John Wiley & Sons and Chapman & Hall, 1951), Chs. VII—X, pp. 142—73. If there is joint production, then the determination of the optimal set of process and therefore of prices according to Taylor's rule cannot be made independently of the demand function.

[18]) See part IV, section C.

The firm's behavior defines the supply and demand functions of firms. In Lange's first model, the demand functions of consumers are defined by utility maximization. The supply and demand functions for the whole market now being defined, the state is to vary prices in accordance with supply and demand. Lange recognized that it had not been proved that this dynamic process would necessarily converge to the equilibrium which corresponded to optimal resource allocation; the process might not converge to anything at all but instead oscillate indefinitely or even diverge explosively.[19]) He suggests that the price-adjustment rules of the state might have to be modified in some way to avoid oscillations, as by taking account of the anticipated effects of price changes on quantities, a point which is illustrated below for the case of increasing returns.[20])

Lange's discussion of the second model, where there is only one utility function, that of a Central Planning Board, does not clearly define the dynamic process.[21]) The meaning seems to be that the Central Planning Board determines its demand functions from its utility function as if it were a consumer and then prices respond to supply and demand, as before. It is not clear whether the Board is subjected to a budget limitation or it simply equates marginal utility to price.

Samuelson's well-known restatement of the Walrasian dynamic system assumes that supply and demand functions are well-defined. His simultaneous dynamic system is then simply the statement that the rate of change of each price is proportional to the difference between demand and supply. Though the essentials of the system are found in Walras and Lange, it is Samuelson who has first made it explicit, and doubtless our reading of the first two is done through the spectacles supplied by Samuelson.

The development of linear programming put renewed emphasis on the problems raised by constant returns in production. In a purely linear economy, where each process operates under constant returns and the utility function of the economic system

[19]) Lange, *op. cit.*, fn 43, pp. 89—90.
[20]) See part IV, parts D, E, and F.
[21]) Lange, *op. cit.*, pp. 90—93.

is linear, the system of computations analogous to the market led to indefinite oscillations, as was shown by Samuelson.[22]) Indeed, this problem was the starting point of the present investigation for the authors.

C. *Formal Statement of the Resource Allocation Model.*

Let there be s commodities and m processes for carrying on production of commodities. Each process may be carried on at different scales, and it is not supposed in general that there are constant returns to scale. Let x_j be the scale of the j^{th} process; for most purposes, x_j may most conveniently be thought of as the amount of output of the j^{th} process, or an index of outputs if the process has more than one. Let $g_{ij}(x_j)$ be the amount of commodity i produced by the j^{th} process when the latter is conducted at scale x_j; a negative value for $g_{ij}(x_j)$ refers to an input.

Among the s commodities we will distinguish a sub-class of n *desired* commodities which enter into final uses. The remaining or *primary* commodities are useful only because they enter into the production of the desired commodities directly or indirectly. Let y_i be the amount of desired commodity i used by the economy for final demands — that is, not used up in one of the productive processes. The economy is assumed to possess a single utility function which has as variables the final demands for the desired commodities — that is, a function,

$$(1) \qquad U(y_1, \ldots, y_n).$$

The total output of commodity i by the productive processes is $\sum_{j=1}^{m} g_{ij}(x_j)$. In addition, some commodities (particularly natural resources and labor) are available in positive quantities without production. Let ξ_i be the amount of commodity i available initially; ξ_i will, of course, be zero for most commodities. A primary commodity for which $\xi_i = 0$ is usually referred to as an *intermediate* commodity, but we shall not need to make a

[22]) In an unpublished memorandum of 1950; see R. Dorfman, P. A. Samuelson, and R. Solow, *Linear Programming and Economic Analysis* (New York, Toronto, and London: McGraw-Hill, 1958), fn. 1, p. 63. An example is given in part II, section D.

distinction between primary and intermediate commodities. In order for a given set of final demands $y_1, \ldots, y_n$ to be feasible, it is necessary that they do not exceed the total available from the productive sector plus the initial availabilities; it is also necessary that the total output of the primary commodities plus that initially available be at least zero. Hence,

$$(2.1) \qquad y_i \leqq \sum_{j=1}^{m} g_{ij}(x_j) + \xi_i \qquad (i = 1, \ldots, n),$$

$$(2.2) \qquad 0 \leqq \sum_{j=1}^{m} g_{ij}(x_j) + \xi_i \qquad (i = n+1, \ldots, s).$$

Here we designate the desired commodities as $1, \ldots, n$ and the primary commodities as $n + 1, \ldots, s$. The resource allocation problem is then to choose $y_1, \ldots, y_n$ and $x_1, \ldots, x_m$ so as to maximize $U(y_1, \ldots, y_n)$ among all sets of variables which satisfy the feasibility conditions (2).

It is to be noted that we have assumed the absence of external economies and diseconomies (as between processes), since the inputs and outputs g_{ij} do not depend upon the scale of any other process than the j^{th}.

We have presented the production sector of the resource allocation model in the form of processes rather than the more usual production functions. In this, we follow Koopmans[23]) who argues persuasively that the latter is a derived concept which already implies some elements of optimization. The formulation in terms of processes is also preferable if the model is interpreted as referring to optimization within a large firm. However, we generalize Koopmans' model by admitting non-linear functions g_{ij}, so that the effects of non-constant returns to scale can be studied.

For future reference, note that the variables $y_1, \ldots, y_n$, $x_1, \ldots, x_m$ are necessarily non-negative from their very definition.

[23]) T. C. Koopmans, "Analysis of Production as an Efficient Combination of Activities," ch. III in *Activity Analysis*, pp. 33—97, especially pp. 33—34.

II. MATHEMATICAL BACKGROUND

A. *Some Notes on Unconstrained Maxima.*

1. Necessary Conditions.

The numbers $\bar{z}_1, \ldots, \bar{z}_p$ form the (*unconstrained*) *maximum* of the (real valued) function $f(z_1, \ldots, z_p)$ if $f(\bar{z}_1, \ldots, \bar{z}_p) \geqq f(z_1, \ldots, z_p)$ for all possible combinations of values of $z_1, \ldots, z_p$. It is permissible to think of the z_i's as being restricted to non-negative values, if appropriate.

For an economic example of unconstrained maximization in the present model, consider the manager of a single process, say the j^{th}, who buys and sells the commodities at prices $p_1, \ldots, p_n$ which are given to him. His profit, then, is

$$\pi_j(x_j) = \sum_{i=1}^{n} p_i g_{ij}(x_j). \tag{3}$$

In a competitive world, the manager seeks to choose x_j to maximize (3). Of course, this is an example where there is only one variable.

We are not primarily interested in unconstrained maxima as such, but a brief discussion of some points connected with them will serve to illustrate developments in the more complicated case of constrained maxima with which we are more concerned.

It should first be noted that an arbitrary function need not have a maximum. For example, in (3), if each of the functions $g_{ij}(x_j)$ is linear, then so is $\pi_j(x_j)$ (recall that the prices p_i are taken as given numbers, not variables as far as the process manager is concerned). If the linear function $\pi_j(x_j)$ has a positive slope, then π_j can be made as large as desired by making x_j sufficiently large, so that no finite value can be designated as the maximum. Even a non-linear function need not have a maximum.

Second, it should be noted that a function may have more than one maximum. An extreme case is a constant, for which all points are maxima; thus if the profit function $\pi_j(x_j)$ is everywhere zero, a situation which arises in linear programming, all values of x_j maximize it.

Third, we have defined a maximum with respect to all per-

missible variations in the variables, sometimes referred to as a *global maximum*. It is frequently useful to use the weaker concept of a *local maximum*, a set of numbers $\bar{z}_1, \ldots, \bar{z}_p$ such that $f(\bar{z}_1, \ldots, \bar{z}_p) \geqq f(z_1, \ldots, z_p)$ for possible values of the z_i's in a small neighborhood of $\bar{z}_1, \ldots, \bar{z}_p$. A global maximum is clearly a local maximum but not, in general, conversely. In this work the term, "maximum," when unqualified, will refer to a global maximum.

If the function f is differentiable and has a local maximum and the variables z_i are unrestricted as to sign, then, as is well known, the partial derivatives $\partial f/\partial z_i = 0$ for all $i = 1, \ldots, p$ when evaluated at the maximum. We shall use the notation f_{z_i} for $\partial f/\partial z_i$ in general and $\bar{f}_{z_i}$ for f_{z_i} evaluated at $\bar{z}_1, \ldots, \bar{z}_p$. Then a necessary condition for a local (and hence for a global) maximum when the z_i's are not restricted as to sign is that $\bar{f}_{z_i} = 0$ $(i = 1, \ldots, p)$.

This last condition is, of course, not sufficient even for a local maximum, let alone a global one. The condition also holds at minima and indeed at some points which are neither maxima nor minima.

2. Concave Functions and Sufficient Conditions.

The condition that the derivatives be zero does become a sufficient condition for a global maximum if the function $f(z_1, \ldots, z_p)$ is restricted to a special class, known as *concave functions*. If there is only one variable, a concave function is one in which the slope is never increasing as the variable increases. A function of one variable is said to be *strictly concave* if the slope is decreasing; such a function is illustrated in Figure 1. A concave function which is not strictly concave differs only in that the graph might have linear segments. It is easy to see graphically that the point $\bar{z}$ at which the derivative is zero is indeed the maximum.

In Figure 1 it can be seen that if we take two points, such as z and z', the part of the graph between the two points lies above the line segment joining them. This property can serve to yield definitions of concavity and strict concavity for functions of any number of variables.

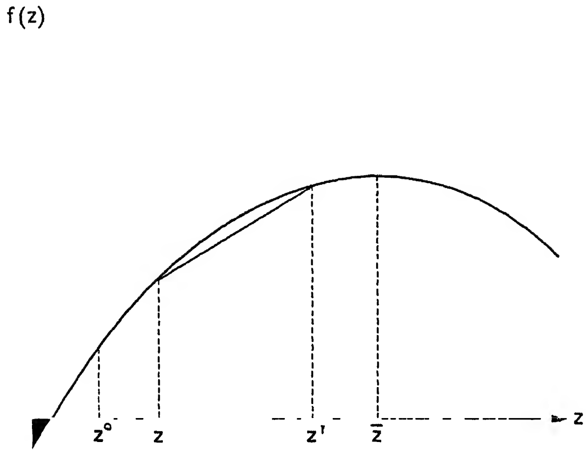

Fig. 1

Definition 1. The (real-valued) function $f(z_1, \ldots, z_p)$ is said to be *concave* if for any two points $(z_1, \ldots, z_p)$ and $(z_1', \ldots, z_p')$ and any third point $(z_1'', \ldots, z_p'')$ such that $z_i'' = \theta z_i + (1-\theta)z_i'$ $(i = 1, \ldots, p)$, where θ is a real number between 0 and 1, $f(z_1'', \ldots, z_p'') \geqq \theta\, f(z_1, \ldots, z_p) + (1-\theta)f(z_1', \ldots, z_p')$. The function $f(z_1, \ldots, z_p)$ is said to be *strictly concave* if the strict inequality holds in the last statement.

Note that a linear function is concave but not strictly concave.

In the case of a single process in a competitive world, where the function (3) is to be maximized, then the condition $d\pi_j/dx_j = 0$ is both necessary and sufficient for maximization if the function $\pi_j(x_j)$ is concave and differentiable. That is, the point of maximum profit is that for which the *marginal profitability* is zero. (If the function $\pi_j(x_j)$ is not concave, then this need not be true; in that case, the point where the marginal profitability is zero might conceivably be a point of minimum profit.) Suppose now that each of the functions $g_{ij}(x_j)$ is concave; then it is easy to see that, since the prices p_i are presumably non-negative, the function $\pi_j(x_j)$ is also concave. Thus the concavity of the functions $g_{ij}(x_j)$ insures that the condition of zero marginal profitability (price equals marginal cost) is equivalent to profit maximization.

What is the meaning of the condition that the functions $g_{ij}(x_j)$ are concave? For simplicity, assume that the process has only one output, say commodity 1, and that the scale of the process is measured by that output so that $g_{1j}(x_j) = x_j$ (note that this function, being linear, is certainly concave). Since the other commodities are inputs, $g_{ij}(x_j)$ is the negative of the input of commodity i for $i > 1$. Since also $g_{ij}(0) = 0$ (for zero output, zero inputs suffice), it is easy to see that concavity of the function $g_{ij}(x_j)$ means that the input of the i^{th} commodity increases at least as rapidly as the output, which is the same as non-increasing returns. Strict concavity would imply diminishing returns, while the requirement of concavity alone permits constant returns.

3. The Gradient Method.

Now consider the problem of finding an unconstrained maximum by a process of successive approximations, starting from some given first approximation, say $z_1^0, \ldots, z_p^0$. One possible procedure is to vary each coordinate separately in such a way as to increase the function $f(z_1, \ldots, z_p)$. Thus, suppose $\partial f/\partial z_i > 0$ at the initial point. Then it is reasonable to increase z_i somewhat; the contrary would be true if $\partial f/\partial z_i < 0$. The same reasoning applies to each coordinate. Thus a reasonable process would call for picking a new point $z_1^1, \ldots, z_p^1$ in such a way that,

(4) $z_i^1 - z_i^0$ has the same sign as $f_{z_i}(z_1^0, \ldots, z_p^0)$ for each $i = 1, \ldots, p$.

The same procedure can be applied again starting with $z_1^1, \ldots, z_p^1$; indeed, such repetition is the essence of an iterative procedure. Thus, at the t^{th} step,

(5) $z_i^{t+1} - z_i^t$ has the same sign as $f_{z_i}(z_1^t, \ldots, z_p^t)$ for each $i = 1, \ldots, p$.

The method specified by (5) is a method of finite differences; mathematically, it is simpler to assume that the process of adjustment takes place continuously rather than in the small steps implied in (5). To effect this, the finite difference $z_i^{t+1} - z_i^t$ should be replaced by the corresponding derivative dz_i/dt, that is, the rate of change of the i^{th} coordinate with respect to (computational) time. The requirement in (5) is then replaced

by the condition

(6) dz_i/dt has the same sign as $f_{z_i}(z_1, \ldots, z_p)$.

Finally, the simplest way to insure that (6) holds is to require[24])

(7) $dz_i/dt = k_i\ (\partial f/\partial z_i) \quad (i = 1, \ldots, p)$,

where k_i is a positive constant. Relation (7) defines a system of differential equations; the solution of this sytem defines each coordinate as a function of time, and this is a description of the adjustment process. We shall refer to (7) as the *gradient method.*[25]) The constant k_i is the *adjustment speed.*

It is easy to see that we can assume $k_i = 1$ without loss of generality simply by changing the units in which z_i is measured. For let $z'_i = a_i z_i$; then $dz'_i/dt = a_i dz_i/dt$, while $\partial f/\partial z'_i = (1/a_i)(\partial f/\partial z_i)$, so that (7) becomes

$$dz'_i/dt = a_i^2 k_i\ (\partial f/\partial z'_i).$$

We can choose a_i so that $a_i^2 k_i = 1$. In this form, with which we shall be mostly concerned, the gradient method becomes

(8) $dz_i/dt = \partial f/\partial z_i$.

For this method to be acceptable, it is necessary that the solution of (8) converge to the maximum values $\bar{z}_1, \ldots, \bar{z}_p$. If we look again at Figure 1, it is intuitively obvious that the process (8) will indeed converge to $\bar{z}$; for if the starting point z^0 is below $\bar{z}$, the rule (8) calls for increasing z up to the point $\bar{z}$ where the derivative is zero but not going beyond. In general, we can make the following statement:

(9) If $f(z_1, \ldots, z_p)$ is a strictly concave function, then the gradient process (8) converges to the maximum point $(\bar{z}_1, \ldots, \bar{z}_p)$.

[24]) $f_{z_i}(z_1, \ldots, z_p)$ and $\partial f/\partial z_i$ have the same meaning.

[25]) See H. B. Curry, "The Method of Steepest Descent for Non-Linear Maximization Problems," *Quarterly of Applied Mathematics*, Vol. 2 (1944), 258–61, who discusses finite difference methods; C. B. Tompkins, "Methods of Steep Descent," ch. 18 in E. F. Beckenbach, ed., *Modern Mathematics for the Engineer* (New York: McGraw-Hill, 1956), pp. 448–79. A brief discussion is also found in Samuelson, *Foundations*, pp. 301–2. Strictly speaking, (7) is only a special case of the gradient method (sometimes called the method of steepest ascent), but since it is the only form with which we shall be concerned, no confusion will result.

(The requirement of strict concavity instead of concavity is not basic, but the statement of (9) would have to be more complicated otherwise.)

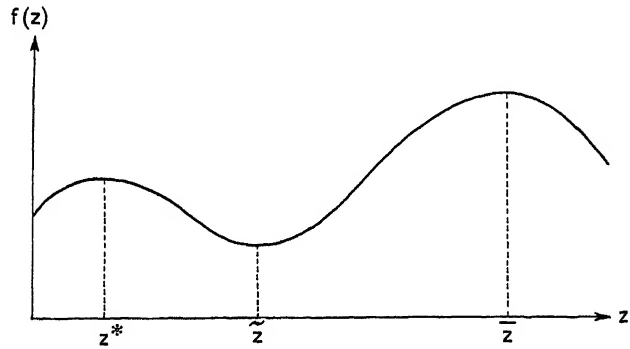

Fig. 2

If $f(z_1, \ldots, z_p)$ is not concave, we might have the situation of Figure 2; there, if the starting point z^0 is below z^* or between z^* and $\tilde{z}$, the process will converge to the local maximum z^* rather than to the global maximum.[26]

4. A Limiting Form of the Gradient Method

We can imagine in equation (7) that some of the adjustment speeds k_i approach infinity. This is equivalent to saying that some of the variables, say $z_1, \ldots, z_r$, are adjusted instantaneously to values which maximize f for given values of the remaining variables $z_{r+1}, \ldots, z_p$. This may be meaningful in situations where the function $f(z_1, \ldots, z_p)$ is of such a simple form with respect to $z_1, \ldots, z_r$ that it is computationally practical to find the maximum with respect to those variables by some fairly direct method. The other variables $z_{r+1}, \ldots, z_p$ are varied in accordance with equations (7), which, by a suitable choice of unit, can be written in the form (8), as has already been explained. The adjustment process then has the form,

(10) $z_1, \ldots, z_r$ maximize $f(z_1, \ldots, z_r, z_{r+1}, \ldots, z_p)$ for given $z_{r+1}, \ldots, z_p$;

[26]) The gradient process for unconstrained maxima will converge in general to some point at which all derivatives are zero.

$$(11) \qquad dz_i/dt = \partial f/\partial z_i, \quad i = r+1, \ldots, p.$$

As the variables $z_{r+1}, \ldots, z_p$ change, the values of $z_1, \ldots, z_r$ which maximize $f(z_1, \ldots, z_p)$ will usually change too,[27] so that all the variables are actually changing in the process. If we assume that the function $f(z_1, \ldots, z_p)$ is concave in all the variables, it is necessarily concave in $z_1, \ldots, z_r$, so that (10) is equivalent to

$$(12) \qquad \partial f/\partial z_i = 0, \quad i = 1, \ldots, r.$$

The adjustment process defined by (11) and either (10) or (12) sounds very reasonable, at least if the equations (12) are computationally practical to solve. The assumption that in a dynamic system some variables adjust slowly while other variables adjust virtually immediately to the first set is a not uncommon one in economics; we shall see its economic interpretation in a resource allocation context later.

This process has one difficulty which has some implications for the structure of the adjustment process in optimal resource allocation. It is possible that for some values of $z_{r+1}, \ldots, z_p$, the function $f(z_1, \ldots, z_p)$ might not have a maximum with respect to $z_1, \ldots, z_r$; that is, $f(z_1, \ldots, z_r, \ldots, z_p)$ might increase indefinitely as one or more of the z_i's $(i = 1, \ldots, r)$ increase to infinity. This is true even if the function $f(z_1, \ldots, z_p)$ is strictly concave. To be sure, if $z_{r+1}, \ldots, z_p$ have the values[28] $\bar{z}_{r+1}, \ldots, \bar{z}_p$, then the function $f(z_1, \ldots, z_p)$ has its maximum value when $z_i = \bar{z}_i$ $(i = 1, \ldots, r)$, so the maximization process is well defined there. In general, the problem does not arise if $z_{r+1}, \ldots, z_p$ are sufficiently close to the maximizing values $\bar{z}_{r+1}, \ldots, \bar{z}_p$, but it might arise otherwise.

We are also implicitly assuming that the maximum, when it exists, is unique.

Subject to these qualifications, however, the limiting form of the gradient process defined by (11) and either (10) or its equivalent (12) has the same satisfactory convergence properties as those of the gradient process (8). Parallel to (9), we can assert,

[27]) I.e., there is a set of functional relations $z_1 = z_1(z_{r+1}, \ldots z_p), \ldots, z_r = z_r(z_{r+1}, \ldots, z_p)$ such that $z_1, \ldots, z_r$ determined from these relations will satisfy (10).

[28]) We recall that f is maximized at the point $z = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_r, \bar{z}_{r+1}, \ldots, \bar{z}_p)$.

(13) if $f(z_1, \ldots, z_p)$ is strictly concave, then in a region sufficiently close to the maximum so that equations (12) are solvable throughout, the process defined by (11) and (12) converges to the maximum point $(\bar{z}_1, \ldots, \bar{z}_p)$.

5. Non-negative Variables.

The preceding discussion has assumed that the variables $z_1, \ldots, z_p$ are unrestricted in range. However, as noted in part I, section C, in our resource allocation problem, we are primarily interested in the case where all variables are required to be non-negative. Consider, for example, the maximization of (3). If the maximizing value of x_j is positive, then, indeed, the derivative has to be zero. Suppose however, $\pi_j(x_j)$ has its maximum at zero; that is, operating the process at any positive level involves a smaller profit (or greater loss) than not operating at all. This implies that at zero, the marginal profitability cannot be positive, but it might be negative, as illustrated in Figure 3.

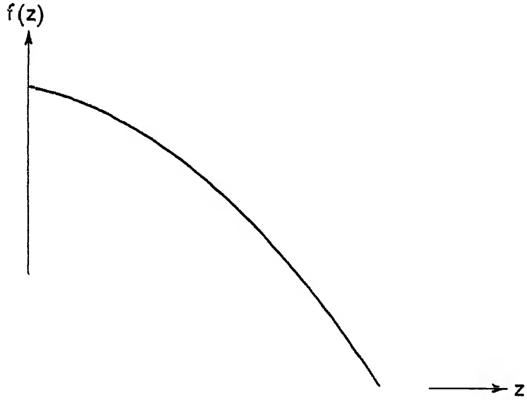

Fig. 3

The same considerations hold when there are several variables. We may thus conclude,

(14) $\bar{f}_{z_i} \leqq 0$ for $i = 1, \ldots, p$; if $\bar{f}_{z_i} < 0$ for some i, then $\bar{z}_i = 0$.

Relations (14) are in general only necessary conditions for a local maximum in general. If the strict inequality never holds, then, as we have seen in paragraph 1 of this section, the point

may be a local minimum. Even if the strict inequality does hold for some components but not for others, a point satisfying (14) may be neither a maximum nor a minimum. Even if it is a local maximum, it need not, of course, be a global one.

However, as in paragraph 2 of this section, relations (14) are a sufficient condition for a global maximum if the function $f(z_1, \ldots, z_p)$ is concave, as illustrated for one variable in Figures 1 and 3.

The gradient method described in (8) also requires some modification when we deal with non-negative variables. If, at some point in the process, one of the variables z_i is zero and if at the same point $\partial f/\partial z_i < 0$, the unmodified gradient method (8) would require z_i to decrease further, that is become negative, which would make no sense if the variable is intrinsically non-negative. This would happen, for example, in Figure 3, if the process has reached the point $z = 0$. To prevent this, we must add to (8) the rule that in this case the variable remains at zero, so that its value does not change, i.e., $dz_i/dt = 0$.

The gradient method then takes the following form:

$$dz_i/dt = \begin{cases} 0 \text{ if } z_i = 0 \text{ and } \partial f/\partial z_i < 0, \\ \partial f/\partial z_i \text{ otherwise.} \end{cases} \tag{15}$$

It is easy to see that in order for the process (15) to stop, that is, for dz_i/dt to equal zero for all i, it is necessary and sufficient that the conditions (14) hold. In other words, the maximum $\bar{z}_1, \ldots, \bar{z}_p$ is the *equilibrium point* of the system of differential equations (15).

In the limiting form of the gradient method, (14) is applicable to the variables $z_1, \ldots, z_r$, in view of (10) applied to the case of non-negative variables. Similarly, for variables $z_{r+1}, \ldots, z_p$, (11) is replaced by (15). The method becomes, then,

$$\partial f/\partial z_i \leqq 0, \text{ with } \partial f/\partial z_i < 0 \text{ only if } z_i = 0, \text{ for } i = 1, \ldots, r; \tag{16}$$

$$dz_i/dt = \begin{cases} 0 \text{ if } z_i = 0 \text{ and } \partial f/\partial z_i < 0, \\ \partial f/\partial z_i \text{ otherwise,} \end{cases} \tag{17}$$

for $i = r + 1, \ldots, p$.

In both forms of the gradient method, convergence is assured

under the same circumstances as in the case of variables unrestricted as to sign.

Finally, for future reference, note that minimization calls for the same rules as maximization, with appropriate changes of sign. Thus, instead of (14), we have for the minimum $\bar{z}$,

(18) $\bar{f}_{z_i} \geqq 0$ for $i = 1, \ldots, p$; if $\bar{f}_{z_i} > 0$ for some i, then $\bar{z}_i = 0$.

Again, (18) is a necessary condition for a minimum in general and a sufficient condition if $f(z_1, \ldots, z_p)$ is convex.[29] The gradient method (15) applied to a minimum becomes,

(19) $$dz_i/dt = \begin{cases} 0 \text{ if } z_i = 0 \text{ and } \partial f/\partial z_i > 0, \\ -\partial f/\partial z_i \text{ otherwise.} \end{cases}$$

B. *Constrained Maxima and Saddle-points.*

The numbers $\bar{z}_1, \ldots, \bar{z}_p$ form the *constrained maximum* of $f(z_1, \ldots, z_p)$ subject to the constraints

(20) $$g_j(z_1, \ldots, z_p) \geqq 0 \qquad (j = 1, \ldots, s),$$

if $f(\bar{z}_1, \ldots, \bar{z}_p) \geqq f(z_1, \ldots, z_p)$ for all combinations of values of $z_1, \ldots, z_p$ which satisfy (20). Again, we may and will restrict ourselves to non-negative values. The resource allocation problem of part I, section C deals with such a constrained maximum; here the z's are interpreted to include the x's and the y's, $f(z_1, \ldots, z_p)$ is identified with $U(y_1, \ldots, y_n)$ (the fact that U does not depend upon all the z's does not create any difficulty), and

(21) $$g_i(z_1, \ldots, z_p) = \begin{cases} \sum_{j=1}^{m} g_{ij}(x_j) + \xi_i - y_i & (i = 1, \ldots, n) \\ \sum_{j=1}^{m} g_{ij}(x_i) + \xi_i & (i = n+1, \ldots, s). \end{cases}$$

When the constraints (20) are in the form of equalities rather than inequalities and the variables are not restricted to be non-negative, the classical method of Lagrange multipliers supplies a necessary condition for a constrained maximum. That is, in order that $\bar{z}_1, \ldots, \bar{z}_p$ maximize $f(z_1, \ldots, z_p)$ subject to

[29]) A *convex* function is one whose negative is concave; that is, f is convex if $-f$ is concave.

$g_j(z_1, \ldots, z_p) = 0$ $(j = 1, \ldots, s)$, it is necessary that there exist $\bar{p}_1, \ldots, \bar{p}_s$ such that

$$\text{(22)} \qquad \begin{aligned} f_{z_i} + \sum_{j=1}^{s} \bar{p}_j \bar{g}_{j,z_i} &= 0 \quad (i = 1, \ldots, p), \\ g_j(\bar{z}_1, \ldots, \bar{z}_p) &= 0 \quad (j = 1, \ldots, s). \end{aligned}$$

Here, $g_{j,z_i} = \partial g_j / \partial z_i$. This condition has been much used in economics.[30] Thus, the theory of consumer's behavior makes use of a special case of (22), where z_i's are the amounts of different commodities purchased, $f(z_1, \ldots, z_p)$ is the utility derived from a given bundle of commodities, and there is just one constraint, the budgetary constraint, which can be written in the form

$$M - \sum_{i=1}^{p} q_i z_i = 0,$$

where q_i is the price of commodity i and M is total income. In this case, $g_{j,z_i} = -q_i$; the Lagrange multiplier $\bar{p}_1$ is interpreted as the marginal utility of income.

Another application of (22), this time using more than one constraint, occurs in Lange's development of the theory of welfare economics.[31]

Equations (22) can be rewritten in an interesting way if we introduce the *Lagrangian* L, defined as,

$$\text{(23)} \quad L(z_1, \ldots, z_p; p_1, \ldots, p_n) = f(z_1, \ldots, z_p) + \sum_{j=1}^{s} p_j g_j(z_1, \ldots, z_p).$$

It is easy to see that,

$$\text{(24)} \qquad \partial L / \partial z_i = f_{z_i} + \sum_{j=1}^{s} p_j g_{j,z_i}, \quad \partial L / \partial p_j = g_j(z_1, \ldots, z_p).$$

In view of (24), the condition (22) which is necessary for a maximum is that,

$$\text{(25)} \qquad \bar{L}_{z_i} = 0, \ \bar{L}_{p_j} = 0, \ i = 1, \ldots, p, \ j = 1, \ldots, s.$$

This form is rather suggestive. If we take, for example, the p_j's

[30] See R. G. D. Allen, *Mathematical Analysis for Economists* (London: Macmillan, 1938) pp. 364—83; Samuelson, *Foundations*, pp. 262—64.

[31] See O. Lange, "The Foundations of Welfare Economics," *Econometrica*, X (1942), 215—28.

as given numbers, with $p_j = \bar{p}_j$ for each j, then (25) shows that the $\bar{z}_i$'s satisfy at least a necessary condition for a maximum or minimum. Before following up this hint, let us reconsider the constrained maximum problem in the form first proposed.

We want to change the problem which leads to (25) in two ways: the z_i's are to be restricted to be non-negative, and the constraints are to be inequalities, as given in (20). Our presentation here is based on the important study by Harold W. Kuhn and Albert W. Tucker.[32] Analogous to (14), it can be shown that the requirement that the z_i's be non-negative requires modifying the first half of (25) to,

$$(26) \qquad \bar{L}_{z_i} \leqq 0 \quad (i = 1, \ldots, p); \text{ if } \bar{L}_{z_i} < 0, \text{ then } \bar{z}_i = 0.$$

Considering the constraints as inequalities requires two modifications. First, suppose that, for some j, $g_j(\bar{z}_1, \ldots, \bar{z}_p) > 0$. Then this particular constraint is ineffective, in that the choices of the z's could have been slightly modified without violating the constraints, but it was not found profitable to do so. Since the solution to the constrained maximization problem should be the same if an ineffective constraint is dropped from the problem completely, we would expect (correctly) that,

$$(27) \qquad \text{if } g_j(\bar{z}_1, \ldots, \bar{z}_p) > 0, \text{ then } \bar{p}_j = 0.$$

The second point is that the $\bar{p}_j$'s must be non-negative. The point of this can be seen by considering a single restraint and giving an economic interpretation to the problem. Let $z_1, \ldots, z_p$ be some economic variables (perhaps outputs), $f(z_1, \ldots, z_p)$ the return from them in utility or money, and $g(z_1, \ldots, z_p)$ the excess supply of some resource, that is, the initial amount available of that resource less the amount needed for the choice of variables $z_1, \ldots, z_p$. The condition that $g(z_1, \ldots, z_p) \geqq 0$ simply amounts to saying that more cannot be used of the resource than is available. Assume, for simplicity, that the $\bar{z}_i$'s are all positive, and that the constraint is effective; then,

$$(28) \qquad \bar{f}_{z_1} + \bar{p}\bar{g}_{z_i} = 0, \; g(\bar{z}_1, \ldots, \bar{z}_p) = 0.$$

[32]) See H. W. Kuhn and A. W. Tucker, "Nonlinear Programming," in J. Neyman, ed., *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (Berkeley and Los Angeles: University of California Press (1952), pp. 481–92.

Consider any movement of the z_i's which will increase the return $f(z_1, \ldots, z_p)$ as compared with the return when the variables take on the values $\bar{z}_1, \ldots, \bar{z}_p$. For example, suppose a small increase in z_1 will increase $f(z_1, \ldots, z_p)$. It must be, then, that such an increase is ruled out as being infeasible in that it violates the constraint $g(z_1, \ldots, z_p) \geqq 0$, for if it were feasible, the $\bar{z}_i$'s could not be the constrained maximum. That is, an increase in z_1 will increase $f(z_1, \ldots, z_p)$ but must decrease $g(z_1, \ldots, z_p)$ below zero. This requires that,

$$\bar{f}_{z_1} > 0,\ g_{z_i} < 0; \tag{29}$$

in view of the first part of (28), we must have $\bar{p} > 0$.

In economic terms, we can imagine that instead of explicitly restraining an economic unit to undertake only feasible policies, we permit any decisions (whether feasible or not) on the z_i's but require it to pay a price p on the amount of the limited resource used. We also let it use any part of the initial supply of the resource or sell any part at the price p. Then the unit's net return can be thought of as $f(z_1, \ldots, z_p) + p\, g(z_1, \ldots, z_p)$, which is the Lagrangian. Then the above asserts that, by choosing p to have the non-negative value $\bar{p}$, the unit will be constrained not to violate the feasibility condition.

If we combine (26), (27), and the preceding discussion, with the aid of (20) and the second half of (24), we have the following theorem, due to Kuhn and Tucker:

> Theorem 1. A necessary condition that $\bar{z}_1, \ldots, \bar{z}_p$ be a constrained local maximum of $f(z_1, \ldots, z_p)$ subject to the constraints $g_j(z_1, \ldots, z_p) \geqq 0$ $(j = 1, \ldots, s)$, with all variables non-negative, is that,
>
> (a) $\bar{z}_i \geqq 0$; $\bar{L}_{z_i} \leqq 0$; if $\bar{L}_{z_i} < 0$, then $\bar{z}_i = 0$ $\quad (i = 1, \ldots, p)$;
>
> (b) $\bar{p}_j \geqq 0$; $\bar{L}_{p_j} \geqq 0$; if $\bar{L}_{p_j} > 0$, then $\bar{p}_j = 0$ $(j = 1, \ldots, s)$.[33]

[33] Strictly speaking, the above theorem is true only if the constraint functions $g_j(z_1, \ldots, z_p)$ satisfy an additional condition, such as that referred to by Kuhn and Tucker as the Constraint Qualification (*op. cit.*, pp. 483—4). A simple condition of this type (used by M. Slater is an unpublished paper) is that there exist some $z_1, \ldots, z_p$ such that $g_j(z_1, \ldots, z_p) > 0$ for all j; in economic terms, that it be possible to choose the economic variables so that there is an excess supply of all resources.

Remark. It is to be stressed that Theorem 1 yields only a *necessary* condition for a *local* maximum. The point is precisely analogous to that which has already been made for unconstrained maxima. Except for the strict inequalities in condition (1), a constrained local minimum would satisfy Theorem 1; even with the inequalities, a point satisfying conditions (a) and (b) might be neither a maximum nor a minimum locally. Further, even if we have a local maximum satisfying the conditions of Theorem 1, there is, in general, no assurance that it will be a global maximum.

Additional conditions on the functions involved are needed to insure that the conditions of Theorem 1 suffice for a global maximum. In view of the corresponding results for unconstrained maxima, it is not surprising to find that conditions (a) and (b) are sufficient for a global maximum if the functions $f(z_1, \ldots, z_p)$ and $g_j(z_1, \ldots, z_p)$ $(j = 1, \ldots, s)$ are all concave functions.

Closely related to this problem is that suggested by the economic interpretation given in the discussion leading up to Theorem 1, namely whether or not the economic unit can be thought of as choosing $z_1, \ldots, z_p$ so as to *maximize* the Lagrangian when the p_j's are set equal to $\bar{p}_1, \ldots, \bar{p}_s$. Condition (a) shows that a *necessary* condition for a maximum must be satisfied (recall condition (14)). It is certainly not always true that the $\bar{z}_i$'s maximize the Lagrangian given the $\bar{p}_j$'s; in economic terms, it is not always true that the optimal allocation of resources will be achieved by maximizing profits even when the prices of resources are properly set.[34]) But there is an important class of cases when in fact we can think of maximizing the Lagrangian with respect to the z_i's. Indeed, we know from earlier discussion (section A.2, this part) that condition (a) of Theorem 1 is sufficient for a maximum if the Lagrangian is a concave function of the z_i's (taking the p_j's as given at $\bar{p}_1, \ldots, \bar{p}_s$). In turn, we can say that this condition is satisfied if $f(z_1, \ldots, z_p)$ and $g_j(z_1, \ldots, z_p)$ $(j = 1, \ldots, s)$ are all concave functions, for then $L(z_1, \ldots, z_p; \bar{p}_1, \ldots, \bar{p}_s)$ is a combination of concave functions with non-negative coefficients, and any such combination is again concave.

[34]) For an emphatic statement of this viewpoint, see Samuelson, *Foundations*, pp. 230—31, 234—35.

(30) If the functions $f(z_1, \ldots, z_p)$ and $g_j(z_1, \ldots, z_p)$ are concave, then the maximum of $L(z_1, \ldots, z_p; \bar{p}_1, \ldots, \bar{p}_n)$ is achieved by setting $z_i = \bar{z}_i$ $(i = 1, \ldots, p)$.

In view of the parallelism between conditions (a) and (b) of Theorem 1, it is natural to ask if the $\bar{p}_j$'s *minimize* the Lagrangian given that $z_i = \bar{z}_i$ for all i. Here, this is clearly true, because the Lagrangian is a linear function of the $\bar{p}_j$'s and all linear functions are convex (as well as concave), so the result follows from (18).

(31) The minimum of $L(\bar{z}_1, \ldots, \bar{z}_p; p_1, \ldots, p_s)$ is achieved by setting $p_j = \bar{p}_j$ $(j = 1, \ldots, s)$.

Statements (30) and (31) suggest the convenience of the following definition as applied to any function depending upon two sets of variables (which we will here regard as non-negative).

Definition 2. The function $\Phi(z_1, \ldots, z_p;\ p_1, \ldots, p_s)$ is said to have a *saddle-point* at $(\bar{z}_1, \ldots, \bar{z}_p;\ \bar{p}_1, \ldots, \bar{p}_s)$ if $\Phi(z_1, \ldots, z_p;\ \bar{p}_1, \ldots, \bar{p}_s)$ has its maximum in $z_1, \ldots, z_p$ at $\bar{z}_1, \ldots, \bar{z}_p$ and $\Phi(\bar{z}_1, \ldots, \bar{z}_p;\ p_1, \ldots, p_s)$ has it minimum in $p_1, \ldots, p_s$ at $\bar{p}_1, \ldots, \bar{p}_n$.

The concept of a saddle-point was used by von Neumann and Morgenstern in connection with the theory of zero-sum two-person games.[35] The definition just given is not the most general possible, but it is sufficient for the present purposes. It is useful to think of a game in which player I chooses p real numbers, $z_1, \ldots, z_p$, player II chooses s real numbers, $p_1, \ldots, p_s$, the two choices being made independently, and the amount paid by player II to player I is given by the function $\Phi(z_1, \ldots, z_p; p_1, \ldots, p_s)$. (A negative value means that player I pays player II.) Clearly player I wishes to maximize the function $\Phi(z_1, \ldots, z_p; p_1, \ldots, p_s)$ with respect to the variables $z_1, \ldots, z_p$ at his control for any given choice of player II's variables, while player II wishes to minimize with respect to his variables.

The discussion can be summarized in the following theorem due to Kuhn and Tucker:

[35]) J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior* (1st ed.; Princeton, New Jersey: Princeton University Press, 1944), p. 95.

Theorem 2. If $f(z_1, \ldots, z_p)$ and $g_j(z_1, \ldots, z_p)$ $(j = 1, \ldots, n)$ are concave functions of the non-negative variables $z_1, \ldots, z_p$, then a necessary and sufficient condition that $\bar{z}_1, \ldots, \bar{z}_p$ maximize $f(z_1, \ldots, z_p)$ subject to the constraints $g_j(z_1, \ldots, z_p) \geqq 0$ $(j = 1, \ldots, s)$ is that there exist numbers $\bar{p}_1, \ldots, \bar{p}_s$ such that the Lagrangian,

$$L(z_1, \ldots, z_p; p_1, \ldots, p_s) = f(z_1, \ldots, z_p) + \sum_{j=1}^{s} p_j g_j(z_1, \ldots, z_p),$$

has $(\bar{z}_1, \ldots, \bar{z}_p; \bar{p}_1, \ldots, \bar{p}_s)$ as a saddle-point with all variables being regarded as non-negative.[36,37])

Theorem 2 has considerable economic interest from a static point of view, i.e., as a characterization of the optimal resource allocation. However, we are here more interested in its implications for a process of successive approximations. In the saddle-point problem, the variables are unconstrained (except that they must be non-negative); hence, it appears possible that a gradient method will make some sense. The equivalence of the saddle-point and constrained maximum problems can then be used to apply the resulting process to the latter problem.

C. *Local Saddle-point Conditions for Constrained Maxima.*

Before discussing the gradient method for saddle-points, we will present some local theorems analogous to Theorem 2. If the concavity conditions are not satisfied, we cannot expect to be able to state simple conditions which could characterize a global maximum or distinguish it from a local maximum. However, it would at least be worthwhile to find necessary and sufficient conditions for a local maximum. Theorem 1 does not satisfy this requirement: it only states necessary conditions.

[36]) Again, the above theorem is true only if the constraint qualification holds; see footnote 33.

[37]) The equivalence relation between saddle-points and constrained maxima was previously studied in the case where the functions $f(z_1, \ldots, z_p)$, $g_j(z_1, \ldots, z_p)$ are linear, that is, in the case of linear programming, by D. Gale, H. W. Kuhn and A. W. Tucker, "Linear Programming and the Theory of Games," ch. XIX in *Activity Analysis*, pp. 317—29, and G. B. Dantzig, "A Proof of the Equivalence of the Programming Problem and the Game Problem," ch. XX, *ibid.*, pp. 330—35. The Kuhn-Tucker theorem given in the text presents a somewhat different form of the relation between the two types of extrema as well as an extension to nonlinear maximand and constraints.

Theorem 2 suggests that a local constrained maximum might be characterized by the local saddle-points of a Lagrangian. This is not true if we take the Lagrangian in the form in which it has been used, but it is true for modifications of the original Lagrangian, of which we present two. First, we will show by an example that a local constrained maximum is not in general a local saddle-point of the Lagrangian. Let the maximand be $f(z_1, z_2) = z_1^2 + z_2^2$ and the constraint $g(z_1, z_2) = 1 - z_1 - z_2 \geqq 0$. It is easy to see that the maximum, for non-negative variables, is attained at two points, (0, 1) and (1, 0). Let us consider the second. The Lagrangian is,

$$L(z_1, z_2; p) = z_1^2 + z_2^2 + p(1 - z_1 - z_2). \tag{32}$$

By Theorem 1, at the maximum, $\bar{L}_{z_1} = 0$ (since $\bar{z}_1 = 1 > 0$), so that $\bar{p} = 2$. If the optimal solution were a local saddle-point, the point (1, 0) would be a local maximum of,

$$L(z_1, z_2; \bar{p}) = z_1^2 + z_2^2 + 2(1 - z_1 - z_2) = (z_1 - 1)^2 + (z_2 - 1)^2. \tag{33}$$

But clearly any change in z_1 would increase (33), so that the point cannot be a local maximum of (33) and therefore not a local saddle-point.

Our program, then, is to search for a modification of the Lagrangian so that a constrained maximum will correspond to a saddle-point. First we remark that no difficulty occurs with respect to the minimization part of Definition 2, as we have already seen in equation (31); that is, the $\bar{p}_j$'s minimize the Lagrangian $L(\bar{z}_1, \ldots, \bar{z}_p; p_1, \ldots, p_s)$ with respect to the p_j's regardless of assumptions about the functions $f(z_1, \ldots, z_p)$, $g_j(z_1, \ldots, z_p)$. This is because the Lagrangian is linear in the p_j's. Our concern therefore is to modify the Lagrangian so that it be locally concave in the z_i's when the p_j's are set equal to their equilibrium values. More precisely, we shall seek to make the Lagrangian *strictly* concave locally in the z_i's; the strict concavity will be important in the applications of the gradient method to be discussed in the next section. We will refer to Lagrangians modified so as to have the desired concavity properties as *concavified Lagrangians.*

Our first method for concavifying Lagrangian is based on

the remark that the constraints in a given maximization problem can be described in different ways. This is analogous to the well-known proposition in consumers' demand theory that monotone transformations of the utility function leave the demands unchanged, the point being that the set of variables which maximizes a function also maximizes any increasing function of it. In the same way, it is possible to transform the constraints without changing the set of values of the variables which satisfy them. Let $\rho_j(u_j)$ be any (real-valued) function of one variable which preserves signs, that is, $\rho_j(u_j)$ is positive, negative or zero according as u_j is positive, negative, or zero, respectively. Let one of the constraints in a maximization problem be that $g_j(z_1, \ldots, z_p) \geqq 0$. Define a new function of the variables $z_1, \ldots, z_p$,

$$(34) \qquad g_j^*(z_1, \ldots, z_p/\rho_j) = \rho_j[g_j(z_1, \ldots, z_p)].$$

In other words, for any given set of values of the variables z_i, the corresponding value of g_j^* is obtained by first computing $g_j(z_1, \ldots, z_p)$, and then computing the value of $\rho_j(u_j)$ when u_j is set equal to the value of $g_j(z_1, \ldots, z_p)$. Since the function $\rho_j(u_j)$ preserves signs, it follows from (34) that

$$(35) \qquad g_j^*(z_1, \ldots, z_p/\rho_j) \geqq 0 \text{ if and only if } g_j(z_1, \ldots, z_p) \geqq 0.$$

The constraints then are unchanged by the transformation. For any sign-preserving functions $\rho_j(u_j)$, any values of the variables $z_1, \ldots, z_p$ which satisfy the original constraints $g_j(z_1, \ldots, z_p) \geqq 0$ also satisfy the transformed constraints $g_j^*(z_1, \ldots, z_p/\rho_j) \geqq 0$, and conversely therefore in particular the values $\bar{z}_1, \ldots, \bar{z}_p$ which maximize $f(z_1, \ldots, z_p)$ under the original constraints are the same as those which maximize it under the transformed constraints. For the transformed problem, the Lagrangian becomes

$$(36) \qquad L^*(z_1, \ldots, z_p; p_1, \ldots, p_s/\rho) = f(z_1, \ldots, z_p) + \sum_{j=1}^{s} p_j g_j^*(z_1, \ldots, z_p/\rho_j)$$

The necessary conditions of Theorem 1 must hold for all the Lagrangians formed by transforming the constraints.

The problem of concavification thus becomes one of choosing among the Lagrangians with transformed constraints a class of Lagrangians for which the concavity property holds. We want however a class which can be defined rather broadly in advance of solving the problem; there is no point in merely showing the existence of a set of functions $\rho_j(u_j)$ for which the Lagrangian (36) is locally strictly concave in the z_i's when $p_j = \bar{p}_j$ for all j if the determination of that set is as hard as the original maximization problem. Fortunately, it is possible to specify a whole class of transforming functions $\rho_j(u_j)$ which depend only on a very general knowledge of the functions involved.

We will present the result in terms of a very specific type of transforming function. Note that since $\rho_j(u_j)$ is negative for u_j negative and positive for u_j positive, it must be increasing at $u_j = 0$. Let us consider the class of functions,

$$\rho_j(u_j) = 1 - (1 - u_j)^{1+\eta_i}, \tag{37}$$

with η_i an even integer which has this property. The following theorem can be demonstrated:

Theorem 3. Under certain regularity conditions, for all even η_j's sufficiently large, a necessary and sufficient condition that $\bar{z}_1, \ldots, \bar{z}_p$ be a local maximum of $f(z_1, \ldots, z_p)$ subject to the constraints $g_j(z_1, \ldots, z_p) \geqq 0$ is that there exist numbers $\bar{p}_1, \ldots, \bar{p}_s$ such that the Lagrangian,

$$L^*(z_1, \ldots, z_p;\ p_1, \ldots, p_s/\eta) = f(z_1, \ldots, z_p) + \sum_{j=1}^{s} p_j \left(1 - [1 - g_j(z_1, \ldots, z_p)]^{1+\eta_j}\right),$$

has $(\bar{z}_1, \ldots, \bar{z}_p;\ \bar{p}_1, \ldots, \bar{p}_s)$ as a local saddle-point, all variables being considered as non-negative. In particular, the function $L^*(z_1, \ldots, z_p;\ \bar{p}_1, \ldots, \bar{p}_s/\eta)$ is locally strictly concave in $z_1, \ldots, z_p$.[38]

Notice that the transformation can be chosen from among a wide class since all η_j's sufficiently large will suffice. The particu-

[38] For a demonstration and more complete spelling out of the regularity conditions, see K. Arrow and L. Hurwicz, "Reduction of Constrained Maxima to Saddle-point Problems," in J. Neyman, ed., *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability* (Berkeley and Los Angeles: University of California Press, 1957), V, 1—20.

lar choice of transformation function (37) is not essential; the basic condition is that the ratio,

$$\rho_j''(0)/\rho_j'(0),$$

be sufficiently large. Another example is $\rho_j(u_j) = 1 - e^{-\eta_j u_j}$.

We will refer to the Lagrangian of Theorem 3 as the *concavified Lagrangian with transformed constraints*. It is in fact a Lagrangian in the ordinary sense applied to a problem which is identical with the original but has the constraints restated.

Before leaving this method of concavification, we note that it can be applied to the situation where the maximand $f(z_1, \ldots, z_p)$ and the constraints $g_j(z_1, \ldots, z_p)$ are linear. In this case, Theorem 2 is applicable, but the Lagrangian is linear in the z_i's, not strictly concave. The latter property, as we have remarked, is desirable in applications of the gradient method to be discussed in the next section. The class of transformation functions which will insure something close to strict concavity of the Lagranian is very wide indeed, including all strictly increasing strictly concave functions. An application to the resource allocation problem in the linear ease will be made in section G, part IV.

We now proceed to an alternative method of achieving a concave Lagrangian, based on a lemma due to Debreu.[39] In this case, the Lagrangian is modified by subtracting a quadratic function of the constraints; we will therefore refer to it as the *concavified Lagrangian with quadratic modification*. First, consider the case in which the constraints are equalities rather than inequalities, that is, the problem is to maximize $f(z_1, \ldots, z_p)$ subject to the constraints $g_j(z_1, \ldots, z_p) = 0$. Then consider the expression,

$$(38) \quad L^\dagger(z_1, \ldots, z_p; p_1, \ldots, p_s/\lambda)$$
$$= f(z_1, \ldots, z_p) + \sum_{j=1}^{s} p_j g_j(z_1, \ldots, z_p) - \lambda \sum_{j=1}^{s} [g_j(z_1, \ldots, z_p)]^2,$$

which differs from the usual Lagrangian only in its last term. It is easy to see that the first-order conditions for a constrained maximum are the same for the Lagrangian with quadratic

[39]) G. Debreu, "Definite and Semidefinite Quadratic Forms," *Econometrica*, Vol. 20 (1952), 296.

modification as for the usual Lagrangian. But it can also be shown, that,[40])

(39) $L^\dagger(z_1, \ldots, z_p;\ \bar{p}_1, \ldots, \bar{p}_s/\lambda)$ is locally strictly concave in $x_1, \ldots, z_p$ for all λ sufficiently large.

The case of constraints which are inequalities can now be covered by introducing new non-negative variables $w_1, \ldots, w_s$, and observing that the condition $g_j(z_1, \ldots, z_p) \geqq 0$ is equivalent to the condition $g_j(z_1, \ldots, z_p) - w_j = 0$. The constraints now being written in the form of equalities, we can apply (39), provided it is observed that the set of variables with respect to which maximization is carried out has been enlarged by the addition of the w_j's.

Theorem 4. Under certain regularity conditions, for all λ sufficiently large, a necessary and sufficient condition that $\bar{z}_1, \ldots, \bar{z}_p$ be a local maximum of $f(z_1, \ldots, z_p)$ subject to the constraints $g_j(z_1, \ldots, z_p) \geqq 0$ if there exist $\bar{p}_1, \ldots, \bar{p}_s$ and $\bar{w}_1, \ldots, \bar{w}_s$ (these numbers are the same for all λ), such that the Lagrangian with quadratic modification,

$$L^\dagger(z_1, \ldots, z_p,\ w_1, \ldots, w_s;\ p_1, \ldots, p_s/\lambda)$$

$$= L(z_1, \ldots, z_p; p_1, \ldots, p_s) - \sum_{j=1}^{s} p_j w_j - \lambda \sum_{j=1}^{s} [g_j(z_1, \ldots, z_p) - w_j]^2,$$

have $(\bar{z}_1, \ldots, \bar{z}_p,\ \bar{w}_1, \ldots, \bar{w}_s;\ \bar{p}_1, \ldots, \bar{p}_s)$ as a local saddle-point, all variables being considered as non-negative. In particular the function $L^\dagger(z_1, \ldots, z_p, w_1, \ldots, w_s; \bar{p}_1, \ldots, \bar{p}_s/\lambda)$ is locally strictly concave in $z_1, \ldots, z_p,\ w_1, \ldots, w_s$.

Remark. When the constraints are equalities, the equilibrium values of the Lagrange multipliers $\bar{p}_j$ may be positive or negative, and the condition $g_j(\bar{p}_1, \ldots, \bar{p}_s) = 0$ is the same as the condition that $\bar{L}_{p_j} = 0$. Since the Lagrangian is linear in the p_j's, this means that for equilibrium values of the z_i's, it is a constant with respect to the p_j's. In particular, one can speak of it as taking a minimum with respect to the p_j's when $p_j = \bar{p}_j$; here

40) See K. Arrow and R. M. Solow, "The Gradient Method for Constrained Maxima Under Weakened Conditions," ch. 10 in K. Arrow, L. Hurwicz, and H. Uzawa, *Studies in Linear and Nonlinear Programming* (Stanford, California: Stanford University Press, 1958), especially section 4. The result is essentially an extension of Debreu's lemma to cover the case of non-negative variables.

the range of variation of the p_j's is unrestricted as to sign. The same statements hold for the Lagrangian with quadratic modification, since the p_j's do not enter the additional quadratic term. It follows that in the saddle-point statement of Theorem 4, one could properly say that the p_j's should be unrestricted as to sign, instead of being non-negative. However, it can be proved that when the constraints are introduced in the special form used there, with a slack variable w_i to transform inequalities to equalities, a saddle-point with respect to non-negative values of the p_j's is also a saddle-point with respect to unrestricted variations.

D. *The Gradient Method for Saddle-points.*

Let us imagine that a game with a saddle-point is played frequently, in fact continuously. At any given trial, player I knows what player II has chosen the previous time. Suppose player I assumes that player II will make the same choice of his variable $p_1, \ldots, p_s$ on the given trial. Then from his point of view, the problem is one of unconstrained maximization. He may then be thought of as starting a gradient process to achieve this maximum, so that this behavior is described by applying (15) to the function $\Phi(z_1, \ldots, z_p; p_1, \ldots, p_s)$, with the variables $p_1, \ldots, p_s$ regarded as given. Thus,

$$dz_i/dt = \begin{cases} 0 \text{ if } z_i = 0 \text{ and } \partial\Phi/\partial z_i < 0, \\ \partial\Phi/\partial z_i \text{ otherwise.} \end{cases} \tag{40}$$

Actually, however, the p_j's are also varying. If player II is thinking along the same lines as player I, he will seek to minimize the payoff function with respect to $p_1, \ldots, p_s$, taking $z_1, \ldots, z_p$ as given. Equation (19) is applicable here, so that,

$$dp_j/dt = \begin{cases} 0 \text{ if } p_j = 0 \text{ and } \partial\Phi/\partial p_j > 0, \\ -\partial\Phi/\partial p_j \text{ otherwise.} \end{cases} \tag{41}$$

We wish to state conditions for the convergence of the gradient method defined by equations (40) and (41). We cannot simply apply the earlier statements about the convergence of the gradient process for an unconstrained maximum. Indeed, if $\Phi(z_1, \ldots, z_p; p_1, \ldots, p_s)$ is strictly concave in the variables $z_1, \ldots, z_p$, taking

$p_1, \ldots, p_s$ as given, then the process (40) would converge if the p_j's remained constant. But in fact the latter variables are simultaneously changing as a result of (41). We may think of an anti-aircraft gun shooting at an airplane. After each shot, the gunner may move the gun so as to correct for the observed error, but meanwhile the airplane is moving away from its previous position and in such a way as to make the error as big as possible. Despite the complexity of the problem, it is possible to show the following result,

(42) if $\Phi(z_1, \ldots, z_p; p_1, \ldots, p_s)$ is *strictly* concave in $z_1, \ldots, z_p$ for each set of values of $p_1, \ldots, p_s$ and convex in $p_1, \ldots, p_s$ for each set of values of $z_1, \ldots, z_p$, then the gradient process described by equations (40) and (41) converges to a saddle-point $(\bar{z}_1, \ldots, \bar{z}_p; \bar{p}_1, \ldots, \bar{p}_s)$ of $\Phi(z_1, \ldots, z_p; p_1, \ldots, p_s)$.[41, 42]

That somewhat stronger conditions than the simple concavity assumptions of Theorem 2 are needed for convergence of the gradient method can be seen from the following example.[43] Suppose there is just one z and one p, and,

$$\Phi(z; p) = z + p - pz.$$

It is easy to verify that the unique saddle-point is $z = 1$, $p = 1$. If we disregard for the moment the non-negativity conditions, the gradient method becomes,

[41] The assumptions of (42) insure that there is a saddle-point and that the z-values must be uniquely defined; it is possible however to have more than one set of p-values.

[42] The fact that the z-values converge to the z-values of a saddle-point has been proved with local assumptions and conclusions by the authors; see K. J. Arrow and L. Hurwicz, "The Gradient Method for Concave Programming I: Local Results," ch. 6 in Arrow, Hurwicz, and Uzawa, *op. cit.* The corresponding theorem in the large was established by H. Uzawa, "The Gradient Method for Concave Programming II: Global Results," ch. 7, *ibid.* That the p-values also converge is shown in K. J. Arrow and L. Hurwicz, "The Gradient Method for Concave Programming III: Further Global Results with Applications to Resource Allocation," ch. 8, *ibid*, referred to below as, "Gradient Method for Resource Allocation," see section 1. For an earlier exposition of the gradient method as applied to constrained maxima and to saddle-points, see K. J. Arrow and L. Hurwicz, "Gradient Methods for Constrained Maxima," *Operations Research*, Vol. 5 (1957), 258—65.

[43] The non-convergence of the gradient method in the linear case was observed by Samuelson; see footnote 22.

$$dz/dt = 1 - p, \ dp/dt = z - 1.$$

The solution of this system of differential equations is,

$$z = 1 + (z_0 - 1) \cos t + (1 - p_0) \sin t,$$
$$p = 1 + (p_0 - 1) \cos t + (z_0 - 1) \sin t, x$$

where (z_0, p_0) is the initial point of the adjustment path. It is easy to see that if z_0 and p_0 are sufficiently close to 1, the solution never becomes negative, so that the above is in fact the solution to the gradient method of (40) and (41) with non-negativity conditions satisfied. Also it is clear that the solution is a periodic function and so never converges to the saddle-point. Instead it cycles endlessly, neither diverging explosively not converging.[44])

E. *The Gradient Method for Constrained Maxima.*

We are now in a position to combine the results of sections B and D. In section B, it was stated that a constrained maximum problem was equivalent to a suitable saddle-point problem. In section D, it was remarked that, under certain condition, a saddle-point could be determined by a gradient process. We have merely to apply (42) to the Lagrangian of Theorem 2.

First, when are the hypotheses of (42) satisfied by the Lagrangian? The Lagrangian $L(z_1, \ldots, z_p; p_1, \ldots, p_s)$ is linear in $p_1, \ldots, p_s$ for any given set of z's and hence certainly convex. Suppose that $f(z_1, \ldots, z_p)$ is strictly concave and the g_j's all concave. Then for any given set of non-negative p_j's, the Lagrangian is a sum of concave functions, one of which is strictly concave; hence the Lagrangian is strictly concave. Thus, the Lagrangian satisfies the hypotheses of (42) if $f(z_1, \ldots, z_p)$ is strictly concave and $g_j(z_1, \ldots, z_p)$ is concave for each j.

To describe the gradient method, we must substitute the Lagrangian $L(z_1, \ldots, z_p; p_1, \ldots, p_s)$ for $\Phi(z_1, \ldots, z_p; p_1, \ldots, p_s)$ n (40) and (41). We can then assert,

Theorem 5. If $f(z_1, \ldots, z_p)$ is strictly concave and $g_j(z_1, \ldots, z_p)$ is concave for each j, then the gradient process defined by,

[44]) This fact was one of the reasons for interest in modified Lagrangians where convergence, rather than endless cycling, could be obtained.

(43) $$dz_i/dt = \begin{cases} 0 \text{ if } z_i = 0 \text{ and } f_{z_i} + \sum_{j=1}^{s} p_j g_{j,z_i} < 0 \\ f_{z_i} + \sum_{j=1}^{s} p_j g_{j,z_i} \text{ otherwise,} \end{cases}$$

(44) $$dp_j/dt = \begin{cases} 0 \text{ if } p_j = 0 \text{ and } g_j(z_1, \ldots, z_p) > 0, \\ -\, g_j(z_1, \ldots, z_p) \text{ otherwise,} \end{cases}$$

converges to a limit $(\bar{z}_1, \ldots, \bar{z}_p;\ \bar{p}_1, \ldots, \bar{p}_s)$, where $(\bar{z}_1, \ldots, \bar{z}_p)$ is the constrained maximum of $f(z_1, \ldots, z_p)$ subject to the restraints $g_j(z_1, \ldots, z_p) \geqq 0$, where the variables z_j are non-negative.

The gradient method of Theorem 5 can be given an economic interpretation along the lines used in section B. If $f(z_1, \ldots, z_p)$ is considered to be the utility derived from setting some variables at levels $z_1, \ldots, z_p$, and $g_j(z_1, \ldots, z_p)$ is the excess supply of resource j, then p_j can be regarded as the price of resource j. Since g_{j,z_i} is the increase in excess supply (decrease in excess demand) due to a unit increase in z_i, we can interpret,

$$-\sum_{j=1}^{n} p_j g_{j,z_i},$$

as the marginal cost attributable to a unit increase in z_i. Hence, the expression,

$$f_{z_i} + \sum_{j=1}^{n} p_j g_{j,z_i},$$

can be interpreted as the difference between marginal utility and marginal cost attributable to a unit increase in z_i. Hence (43) is an instruction to increase z_i if the marginal utility exceeds the marginal cost and decrease it otherwise, with the proviso that the activity level z_i cannot be decreased below zero. Equation (44) has even a simpler interpretation, to increase price if the excess supply is negative (i.e., if demand exceeds supply) and decrease it otherwise, but not below zero. The theorem then asserts that the process just described will eventually converge to the activity levels which yield the highest utility subject to the conditions that demand never exceed supply for any commodity. The process will also define a corresponding set of equilibrium prices.

Theorem 5 is not in present form applicable to the resource allocation model of part I, section C even under the strongest reasonable concavity assumptions. Somewhat stronger theorems are presented in part III, section D.

The proposition contained in (42) can be applied locally to the concavified Lagrangians introduced in section C. Theorems 3 and 4 then yield the following corresponding results.

Theorem 6. If $\bar{z}_1, \ldots, \bar{z}_p$ is a local maximum of $f(z_1, \ldots, z_p)$ subject to the constraints $g_j(z_1, \ldots, z_p) \geqq 0$ and certain regularity conditions are satisfied, then the gradient method applied to the concavified Lagrangian with transformed constraints $L^*(z_1, \ldots, z_p;\ p_1, \ldots, p_s/\eta)$ with η_j's sufficiently large, converges to a saddle-point $(\bar{z}_1, \ldots, \bar{z}_p;\ \bar{p}_1, \ldots, \bar{p}_s)$ if the initial approximation is sufficiently close.

Theorem 7. If $\bar{z}_1, \ldots, \bar{z}_p$ is a local maximum of $f(z_1, \ldots, z_p)$ subject to the constraints $g_j(z_1, \ldots, z_p) \geqq 0$ and certain regularity conditions are satisfied, then the gradient method applied to the concavified Lagrangian with quadratic modification, $L^\dagger(z_1, \ldots, z_p, w_1, \ldots, w_s;\ p_1, \ldots, p_s/\lambda)$ with λ sufficiently large, converges to a saddle-point $(\bar{z}_1, \ldots, \bar{z}_p, \bar{w}_1, \ldots, \bar{w}_s;\ \bar{p}_1, \ldots, \bar{p}_s)$ if the initial approximation is sufficiently close.

F. *The Price-Adjustment Method.*

As in section A.4, we may consider a modification of the gradient method of the previous section in which some of the variables are assumed to adjust inifinitely rapidly, or, in other words, to be at all points in the adjustment process at an optimal position given the values of the other variables. For the gradient method described by (43—44), we consider the variant where the z_i's are supposed to maximize the Lagrangian for any given set of p_j's, while the p_j's continue to be adjusted in accordance with (44). In symbols, (43) is replaced by,

$$(45)\quad z_1, \ldots, z_p \text{ maximize } f(z_1, \ldots, z_p) + \sum_{j=1}^{s} p_j g_j(z_1, \ldots, z_p)$$

for given $p_1, \ldots, p_s$.

(44) and (45) together constitute a dynamic system. This system is closer to the usual supply-and-demand model than

the system of the preceding section. In the present system, for any given set of prices, the activity levels optimal for that set of prices are determined. These determine, in turn, the supply and demand of each commodity, and the prices then move in a direction determined by the difference of supply and demand. In the system of the last section, the activity levels were not at an optimal level but only being varied so as to increase the difference between utility and costs.

As noted in section A.4, any system which depends upon instantaneous optimization with respect to some variables runs the risk of being undefined. At any point in the adjustment process where the prices still differ from equilibrium, some of the optimal activity levels might be infinite. With this qualification, however, the *price-adjustment process* defined by (44) and (45) converges satisfactorily.

Theorem 8. If $f(z_1, \ldots, z_p)$ is strictly concave and $g_j(z_1, \ldots, z_p)$ is concave for each j, then the price-adjustment process defined by (44) and (45) converges in a region sufficiently close to equilibrium so that (45) is always well-defined.

III. CONDITIONS FOR VALIDITY OF THE NATURAL MARKET MECHANISMS

In this part, we will state the assumptions about the resource allocation model of part I, section C, for which the global static and dynamic characterizations of an optimum given in Theorem 2, 5, and 8 of part II can be applied in a straightforward fashion. The resulting criteria have natural economic interpretations. For the static characterization, the assumptions needed are that all the functions involved are concave; for the determination of dynamic processes which converge to an optimum, somewhat stronger conditions are needed.

A. *Static Characterization of the Optimal Allocation.*

We will first make the following two assumptions about the utility sector, respectively, in the resource allocation model of Part. I section C.

$(U-C)$ The utility function $U(y_1, \ldots, y_n)$ is a concave function.
$(P-C)$ The functions $g_{ij}(x_j)$ are concave functions.

Assumption $(P - C)$ is a straightforward statement of non-increasing returns to scale. The status of assumption $(U - C)$ is slightly more complicated. The ordinal point of view would deny meaning to any statement about a utility function which is not invariant under monotonic transformation. Concavity is not a property with the desired invariance; indeed, it implies diminishing (more strictly, non-increasing) marginal utility for each commodity, a well-known shibboleth for distinguishing the cardinalists from the ordinalists.

The usual ordinalist assumption is that the indifference surfaces are convex to the origin. An alternative formulation can be made in terms of the following definitions:

Definition 3. A function $f(z_1, \ldots, z_p)$ is said to be *quasi-concave* if for any two distinct points $(z_1, \ldots, z_p)$ and $(z_1', \ldots, z_p')$ and any third point $(z_1'', \ldots, z_p'')$ such that $z_i'' = \theta z_i + (1 - \theta) z_i'$ $(i = 1, \ldots, p)$, where $0 < \theta < 1$, $f(z_1'', \ldots, z_p'')$ is at least as great as the smaller of the two numbers $f(z_1, \ldots, z_p)$ and $f(z_1', \ldots, z_p')$. The function is said to be *strictly quasi-concave* if the strict inequality holds in the last statement.

For the moment, we will not use the last part of the definition. It is easy to see geometrically that an indifference map which possesses a quasi-concave utility indicator indeed statisfies the usual convexity assumptions.[45]) This property further is invariant under monotone transformations of f.

Now it has been shown that under certain weak regularity conditions, there exists for any quasi-concave function a monotone transform which is concave.[46]) Such a function can, of

[45]) Despite the importance of this assumption in consumer's demand theory, little attention has been given to its justification. For an argument due to T. C. Koopmans, see K. J. Arrow, "An Extension of the Basic Theorems of Classical Welfare Economics," in *Proceedings of the Second Berkeley Symposium*, pp. 529—30; T. C. Koopmans, *Three Essays on the State of Economic Science* (New York, Toronto, and London: McGraw-Hill, 1957), pp. 26—28. See also W. M. Gorman, "Convex Indifference Curves and Diminishing Marginal Utility," *Journal of Political Economy*, LXV (1957), 40—50.

[46]) See W. Fenchel, *Convex Cones, Sets, and Functions* (Department of Mathematics, Princeton University, 1953) (mimeographed), pp. 115—37; B. de Finetti, "Sulle stratificazaioni convesse," *Annali di Matematica Pura e Applicata*, vol. 30 (1949), 173—83.

course, equally well serve as a utility indicator. The significance of $(U - C)$, then, is that among the infinitely many utility functions which represent the indifference map of the economic unit, we can choose (at least) one which is concave.

The problem is to characterize the maximization of the utility function, subject, to the constraints (2), with respect to the variables $y_1, \ldots, y_n, x_1, \ldots, x_m$. It is easy to verify from Definition 1 that a function which is concave with respect to some members of a set of variables and where the other variables do not enter at all is concave with respect to all of them. From $(U - C)$, then, $U(y_1, \ldots, y_n)$ can be regarded as a concave function of all the variables $y_1, \ldots, y_n, x_1, \ldots, x_m$. Similarly, $g_{ij}(x_j)$ are concave functions of the same variables. The function $-y_i$, is a linear and therefore concave function of all these variables. If we let

$$(46)\quad g_i(y_1, \ldots, y_n, x_1, \ldots, x_m) = \sum_{j=1}^{m} g_{ij}(x_j) - y_i + \xi_i \ (i = 1, \ldots, n),$$

$$(47)\quad g_i(y_1, \ldots, y_n, x_1, \ldots, x_m) = \sum_{j=1}^{m} g_{ij}(x_j) + \xi_i \ (i = n+1, \ldots, s)$$

then each of the functions g_i is a sum of concave functions and therefore itself concave. (Note that we write g_i as a function of all the y-variables, though in fact g_i depends only on y_i if $1 \leqq i \leqq n$ and does not depend on any of the y-variables if $i > n$.)

With the aid of (46) and (47), (2.1) and (2.2) can be rewritten

$$(48)\qquad g_i(y_1, \ldots, y_n, x_1, \ldots, x_m) \geqq 0 \quad (i = 1, \ldots, s).$$

The function g_i is the excess of supply (both produced and natural) over demand (both final and interindustry). It will be referred to as the *excess supply*.

The problem of maximizing $U(y_1, \ldots, y_n)$ subject to the constraints (48) now satisfies the conditions of Theorem 1; the variables $y_1, \ldots, y_n, x_1, \ldots, x_m$ correspond to $z_1, \ldots, z_p$, the function $U(y_1, \ldots, y_n)$ to $f(z_1, \ldots, z_p)$, and the functions g_i to the functions g_j. It can therefore be asserted that:

(49) a necessary and sufficient condition that $\bar{y}_1, \ldots, \bar{y}_n, \bar{x}_1, \ldots, \bar{x}_m$ maximize $U(y_1, \ldots, y_n)$ subject to the constraints

(48) is that there exist numbers $\bar{p}_1, \ldots, \bar{p}_s$ such that the Lagrangian

$$L(y_1, \ldots, y_n, x_1, \ldots, x_m; p_1, \ldots, p_s)$$
$$= U(y_1, \ldots, y_n) + \sum_{i=1}^{s} p_i g_i(y_1, \ldots, y_n, x_1, \ldots, x_m),$$

has $(\bar{y}_1, \ldots, \bar{y}_n, \bar{x}_1, \ldots, \bar{x}_m; \bar{p}_1, \ldots, \bar{p}_s)$ as a saddle-point. (Here, p_i is the Lagrange multiplier associated with the constraint $g_i \geqq 0$.)

To simplify notation, we shall write y for $(y_1, \ldots, y_n)$ and similarly with the other sets of variables.

From its definition, the saddle-point criterion contains two statements:

(50.1) the function $L(y, x; \bar{p})$ (i.e., the Lagrangian with the multipliers replaced by their saddle-point values) attains its maximum as a function of y and x when[47]) $y = \bar{y}$ and $x = \bar{x}$;

(50.2) the function $L(\bar{y}, \bar{x}; p)$ attains its minimum at $p = \bar{p}$.

We will now give some economic interpretation to the statements (50) by considering them in more detail. First, consider (50.2); the interpretation has already appeared in the discussion leading up to Theorem 2. The function $L(\bar{y}, \bar{x}; p)$ is linear in p. A mechanical application of (18) in part II, section A.5, leads to the following results:

(51) $$g_i(\bar{y}, \bar{x}) \geqq 0$$

(52) $$\text{if } g_i(\bar{y}, \bar{x}) > 0, \text{ then } \bar{p}_i = 0.$$

(51) is a repetition of the feasibility conditions (48) or (2). (52) adds the condition that for any commodity for which supply exceeds demand at the optimum the associated Lagrange multiplier is zero.

To draw the implications of (50.1), we will replace in the Lagrangian g_i by its definition as given in (46) and (47).

[47]) $y = \bar{y}$ is an abbreviation for the set of equalities $y_1 = \bar{y}_1$, $y_2 = \bar{y}, \ldots$, $y_n = \bar{y}_n$. Similar notation is used for other sets of variables.

(53)
$$\begin{aligned} L(y, x; \bar{p}) &= U(y_1, \ldots, y_n) + \sum_{i=1}^{n} \bar{p}_i \left[\sum_{j=1}^{m} g_{ij}(x_j) - y_i + \xi_i\right] \\ &\quad + \sum_{i=n+1}^{s} \bar{p}_i \left[\sum_{j=1}^{m} g_{ij}(x_j) + \xi_i\right] \\ &= \left[U(y_1, \ldots, y_n) - \sum_{i=1}^{n} \bar{p}_i y_i\right] \\ &\quad + \sum_{j=1}^{m} \left[\sum_{i=1}^{s} \bar{p}_i g_{ij}(x_j)\right] + \sum_{i=1}^{s} \bar{p}_i \xi_i . \end{aligned}$$

The Lagrangian has now been expressed as the sum of $(m+2)$ terms. The first depends on $y_1, \ldots, y_n$, but not on any of the x_j's. Each of the next m is of the form,

$$\sum_{i=1}^{s} \bar{p}_i g_{ij}(x_j) = \bar{\pi}_j(x_j),$$

say, and depends only on one variable, the scale x_j of the j^{th} process. Finally, the last term is a constant. To maximize a sum of functions each depending on a different set of variables involves only maximizing each of them separately. Condition (50.1) can then be written,

(54.1) the function $U(y_1, \ldots, y_n) - \sum_{i=1}^{n} \bar{p}_i y_i$ attains its maximum with respect to $y_1, \ldots, y_n$ at $y_i = \bar{y}_i$ $(i = 1, \ldots, n)$;

(54.2) for each j, the function $\bar{\pi}_j(x_j)$ attains its maximum at $x_j = \bar{x}_j$.

The economic and institutional interpretation of the above criteria, particularly (52) and (54), as is by now familiar, requires identifying the Lagrange multiplier associated with a commodity restraint as the price of that commodity. Thus p_i may be thought of as the price of commodity i. Condition (52) then states that the equilibrium price is zero for any commodity for which there is an excess of supply over demand at equilibrium. Condition (54.1) says that the final demands are to be chosen so as to maximize the difference between their utility and their cost (at equilibrium prices).

In discussing (54.2), first note that, by definition, $\bar{\pi}_j(x_j)$ is the profit (calculated at equilibrium prices) obtained by running

the j^{th} process at scale x_j. Then (54.2) states that each process should be run at a scale which will yield maximum profit.

Conditions (54), particularly, bring out the possibility of decentralization through the price system. Given the set of equilibrium prices, the choice of the final demands can be made separately from the choices of the process scales, and, further, the latter can be made separately from each other. Let us imagine that a *manager* is appointed for each process and that a *helmsman* is charged with choosing final demands.[48]) For any given set of prices, each manager is instructed to choose that scale which will lead to a maximum of the profits,

$$\sum_{i=1}^{n} p_i g_{ij}(x_j) = \pi_j(x;\ p). \tag{55}$$

At the same time, the helmsman is instructed to determine the level of the final demands so as to maximize the difference between utility and costs, $U(y_1, \ldots, y_n) - \sum_{i=1}^{n} p_i y_i$. For each desired commodity i, the choice of x_j by each manager determines an output (possibly negative) $g_{ij}(x_j)$ by the j^{th} process, while the helmsman's decision includes one for y_i. Thus net demand for the desired commodity i for final and intermediate use is,

$$y_i - \sum_{j=1}^{m} g_{ij}(x_j);$$

for the equilibrium prices, conditions (51) and (52) require that this net demand does not exceed the initial supply ξ_i and that, if the two are unequal, the price $\bar{p}_i$ must be zero (if the optimum can be reached without using the full initial supply, then the commodity is a free good). Similarly, for a primary commodity i, the net demand,

$$-\sum_{j=1}^{m} g_{ij}(x_j),$$

must be compared with the initial supply ξ_i; at equilibrium, the net demand must not exceed the initial supply, and the price $\bar{p}_i$ must be zero if there is an excess supply.

[48]) We follow the terminology of Koopmans, "Production as an Efficient Combination . . ," pp. 93—95, as developed for the linear case; however, the function of the helmsman is somewhat different here.

The elements of decentralization here are clear. For a given set of prices, a process manager need know only the prices and the technology of his own process in order to arrive at the optimal level for his process. The helmsman need only know the prices of the desired commodities and the utility function. Finally, the equilibrium on each market may be checked separately; for any given market, the test requires knowing only the net demand, which is an aggregate of many individual decisions, the initial supply, and the price.

Theorem 9. If $(U - C)$ and $(P - C)$ hold, then $\bar{y}_1, \ldots, \bar{y}_n$, and $\bar{x}_1, \ldots, \bar{x}_m$ are final demands $y_1, \ldots, y_n$ and process levels $x_1, \ldots, x_n$ which maximize the utility function $U(y_1, \ldots, y_n)$ subject to the feasibility constraints if there exist prices $\bar{p}_1, \ldots, \bar{p}_s$ such that the following conditions are satisfied:

(a) $\bar{y}_1, \ldots, \bar{y}_n$ maximize the difference, $U(y_1, \ldots, y_n) - \sum_{i=1}^{n} \bar{p}_i y_i$, between utility and costs;

(b) for each process j, $\bar{x}_j$ maximizes the profit, $\sum_{i=1}^{s} \bar{p}_i g_{ij}(x_j)$;

(c) for each desired commodity i, the aggregate excess of supply over demand, $\sum_{j=1}^{m} g_{ij}(\bar{x}_j) + \xi_i - y_i$, is non-negative; if positive, then the price $\bar{p}_i$ is zero.

(d) similarly, for each primary commodity i, the aggregate excess of supply over demand, $\sum_{j=1}^{m} g_{ij}(\bar{x}_j) + \xi_i$, is non-negative; if positive, then the price $\bar{p}_i$ is zero.

It may be noted that there may be more than one maximum in (a) or (b). This will usually be the case in (b) if the functions g_{ij} are linear, as in the linear programming case. In general, in this case, a process manager will find a range of scales, each of which achieves the possible profit, but only some scales will in fact be optimal for the economy. The choice by the manager of a scale which is not optimal for the economy as a whole will be revealed by a violation of one of the conditions (c) or (d). The validity of Theorem 9 is not affected by this remark, but

the force of the decentralization argument is somewhat weakened under these conditions.

B. *An Alternative Static Characterization of the Optimal Allocation.*

Theorem 9 as a characterization of an optimal allocation is, of course, closely related to the criteria of welfare economics, as given by Hotelling, Lange, Koopmans, Allais, Lerner, and Bergson.[49] However, condition (a) is somewhat different from the analogous characterization of Pareto optima when there are many consumers. In the latter case, each consumer is faced with a constrained maximization problem, that of maximizing the utility function subject to a budget restraint. Since we are here treating the special case where there is only one consumer, there should be and is an analogous theorem, with (a) replaced by a constrained maximization.

Let $\bar{M} = \sum_{i=1}^{n} \bar{p}_i \bar{y}_i$, the total expenditure of the helmsman at equilibrium. Suppose he is now told to maximize utility subject to the constraint that his expenditures be $\bar{M}$, i.e.,

$$\sum_{i=1}^{n} \bar{p}_i y_i = \bar{M}. \tag{56}$$

Since the result of a maximization is not affected by changing the maximand by a constant, the problem is equivalent to maximizing $U(y_1, \ldots, y_n) - \bar{M} = U(y_1, \ldots, y_n) - \sum_{i=1}^{n} \bar{p}_i y_i$ subject to the constraint (56). But we already know that the unconditional maximum for the last maximization is $y_i = \bar{y}_i$ $(i = 1, \ldots, n)$. Since this solution satisfies the constraint (56), it must also be the constrained maximum. Hence (a) is equivalent to,

(57) $\bar{y}_1, \ldots, \bar{y}_n$ maximize $U(y_1, \ldots, y_n)$ subject to the constraint (56).

[49]) See H. Hotelling, "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates," *Econometrica*, Vol. 6 (1938), 242—67; Lange, "Foundations, ..." *op. cit.*; Koopmans, ch. 1 in *Three Essays, op. cit.*; M. Allais, *Traité d'Economie Pure* (Paris: Imprimerie Nationale, 1943) III, 604—82; Lerner, *op. cit.*; A. Bergson (Burk), "A Reformulation of Certain Aspects of Welfare Economics," *Quarterly Journal of Economics*, V III (1938), 310—334.

So far, this is not very interesting since the condition (57) requires a knowledge of $\bar{M}$. Now let us recall the homogeneity properties of demand and supply functions derived from utility — and profit — maximization. The behavior described in (57) is that of the consumer in the usual theory of demand. If $\bar{M}$ is changed to any value M and the prices changed in the same proportion, the choice of the y_i's is unaffected. At the same time, the choice of x_j is unaffected by a change of all prices in the same proportion. Hence, if $\bar{M}$ is replaced by any number M, we can choose the price $\bar{p}_i$ so that (57) and (b) of Theorem 9 are satisfied. In (c) and (d), the prices enter directly only in the form of conditions for zero prices; but a change of all prices in the same proportion leaves the zero prices unaltered. We can state,

Theorem 9′. Theorem 9 remains valid if (a) is replaced by,

(a′) $\bar{y}_1, \ldots, \bar{y}_n$ maximize $U(y_1, \ldots, y_n)$ subjects to the budget restraint, $\sum_{i=1}^{n} \bar{p}_i y_i = M$, where M can be chosen arbitrarily.

The constrained maximum (a′) is, as usual in consumers' demand theory, invariant under monotonic transformations of the utility function. It can thus be shown that Theorem 9′, unlike Theorem 9, remains valid if the utility function is to be merely quasi-concave, rather than concave.

C. *The Economic Meaning of the Gradient Method.*

We will now apply the gradient methods of section E, part II to the resource allocation model of part I, section C. The formulation is parallel to the application of Theorem 2 as given in section A of this part. The differential equations (43) and (44) then become,

$$(58) \qquad dy_i/dt = \begin{cases} 0 \text{ if } y_i = 0 \text{ and } (\partial U/\partial y_i) - p_i < 0, \\ (\partial U/\partial y_i) - p_i \text{ otherwise;} \end{cases}$$

$$(59) \qquad dx_j/dt = \begin{cases} 0 \text{ if } x_j = 0 \text{ and } d\pi_j/dx_j < 0, \\ d\pi_j/dx_j \text{ otherwise;} \end{cases}$$

$$(60) \qquad dp_i/dt = \begin{cases} 0 \text{ if } p_i = 0 \text{ and } g_i > 0, \\ -g_i \text{ otherwise.} \end{cases}$$

In (59), π_j is the profit evaluated at current prices, as defined in (55).

These equations can be obtained by directly substituting the particular maximand and constraints of the resource allocation problem into the general gradient method. It is perhaps more illuminating to notice that (58) and (59) are direct dynamic counterparts of the conditions (a) and (b) of Theorem 9. Once the problem of the helmsman has been stated as an *unconstrained* maximization, it is natural to transform the static condition of finding the maximum into the gradient method for approaching it; this is precisely (58) (compare section A.3, part II). Equation (59) has precisely the same relation to condition (b) as (58) has to (a).

But of course we now assume that the maximization processes of (58) and (59) are being applied at any set of prices taken as given to the helmsman and the managers, not merely at the equilibrium set of prices. As the final demands y_i and process scales x_j are varied in accordance with (58) and (59) respectively, the prices which the helmsman and the process managers take as given are themselves being varied in accordance with (60). The meaning of (60) has already been explored in part II, section E.

Thus we see that the gradient method may be given the following institutional interpretation. The helmsman, taking the prices of desired commodities as given, changes each final demand at a rate equal to the difference between marginal utility and price, except that if the final demand for any commodity is zero and the marginal utility is less than the price, the final demand remains at zero (since negative final demands have no meaning). Hence the final demand for a commodity is increased if marginal utility exceeds price and, if not already zero, decreased if marginal utility is less than price. For each process, the manager, taking all prices as given, changes the scale of his process at a rate proportional to its marginal profitability, except that if the scale is zero and the marginal profitability negative, the scale remains at zero. Thus a process increases in scale if the marginal profitability of expansion is positive and decreases in the opposite case if not already zero.

The choice by a manager of his process scale determines the output or input of each commodity in the process. For any one

commodity, total the outputs (taking inputs as negative outputs) for all processes, add the initial supply ξ_i, and, in the case of a desired commodity, subtract the final demand determined by the helmsman. The result is the *excess supply*, denoted by g_i, of the desired and primary commodities respectively. We may imagine for each commodity a *custodian* (again in Koopmans's terminology) who varies the price of the commodity at a rate proportional but opposite in sign to the excess supply, with the qualification that if the price is zero and the excess supply positive the price remains at zero. This instruction is the well-known "law of supply and demand": the price of a commodity rises if demand exceeds supply, falls in the opposite case.

The instructions to the helmsman, the managers, and the custodians represent a high degree of decentralization of the information-gathering and decision-making functions. The helmsman need only know the utility function and the prices of desired commodities to carry out his rule (58). The manager of a process need know only its technology, as represented by the functions $g_{ij}(x_j)$, and the prices of the commodities entering it. A custodian need only know the aggregate difference between demand and supply on his market (not the offers and demands of individual processes or of the helmsman) and the price of his commodity.

D. *The Conditions for Validity of the Gradient Method.*

The solution of the system of differential equations (58—60) will not converge for any arbitrary set of functions $U(y_1, \ldots, y_n)$ and $g_{ij}(x_j)$. Not even the concavity of all these functions suffices. As was seen in part II, section D, if all the functions involved are linear, the gradient method will lead to indefinite oscillations in some or all of the variables with no convergence to the optimum solution.

Theorem 5 supplies a sufficient condition for convergence in the general constrained maximization problem but one that, unfortunately, is not applicable to the resource allocation problem. The reason for inapplicability is that the maximand $U(y_1, \ldots, y_n)$ does not contain all the variables with respect to

which it is maximized and hence cannot be strictly concave in all the variables.

To obtain the desired results we will now strengthen assumption $(U - C)$ to,

$(U - SC)$ $U(y_1, \ldots, y_n)$ is strictly concave in $y_1, \ldots, y_n$.

The justification for $(U - SC)$ is similar to that for $(U - C)$. In consumers' demand theory it is usual to assume not merely that the indifference surfaces are convex to the axes but also that they are not linear or planar and have no linear segments in them. This requirement insures that the demand functions are well-defined (i.e., single-valued) since, for any budget plane, the indifference surface tangent to it has only one point in common with it. We may state this as a requirement on the utility indicator in the terminology of Definition 3.

$(U - SQC)$ $U(y_1, \ldots, y_n)$ is a strictly quasi-concave function.

Under suitable regularity conditions[50]), if $(U - SQC)$ holds, there is a monotonic transform which is strictly concave. Therefore assumption $(U - SC)$ means essentially that a suitable one of the utility indicators has been chosen in the definition of condition (58) of the gradient method.

While assumption $(U - SC)$ is not sufficiently strong to insure that all the variables will converge to their optimal values when the gradient method is applied, it is sufficient to insure that the most important ones do.

Theorem 10. If $(U - SC)$ and $(P - C)$ hold, and the variables y_i, x_j, p_i are varied in accordance with the gradient method defined by equations (58—60), then, for each desired commodity i, the final demand y_i and the price p_i converge to their optimal values $\bar{y}_i$ and $\bar{p}_i$ $(i = 1, \ldots, n)$. The process scales x_j and the prices of the primary commodities p_i $(i = n + 1, \ldots, s)$ may oscillate indefinitely but cannot diverge.

Further, for any desired commodity for which the optimal final demand is positive, the difference between supply and demand approaches zero. In symbols, for any i for which $\bar{y}_i > 0$,

$$\lim_{t \to \infty} [y_i - \sum_{j=1}^{m} g_{ij}(x_j) - \xi_i] = 0.$$

[50]) See footnote 46.

Although Theorem 10 assures convergence in the most important variables, it is not completely satisfactory. The scales of the processes must, in the limit, supply the equilibrium final demands, but it is not guaranteed that they will satisfy the feasibility constraints with respect to primary commodities. The process scales may instead oscillate indefinitely in such a way that the demand by processes for a primary commodity oscillates indefinitely about the initial supply. Examples can be given to show that this possibility is not ruled out by any assumption thus far made.

As might be expected, the possibility of oscillations is connected with linearity of the production processes. We can make the following

Remark. Even if some of the processes are linear, oscillations in the process scales can occur only "by accident", in the sense that special relations must hold among the input-output coefficients which define the different linear processes.

It follows that the gradient method in general provides a satisfactory solution of the resource allocation problem under the assumptions of Theorem 10.

In any case, the absence of linearity is a sufficient condition for convergence in all the variables. We need to assume a sharpened version of $(P - C)$; instead of merely postulating non-increasing returns, we will wish to assume strictly diminishing returns. In mathematical terms, this means that the functions $g_{ij}(x_j)$ defining a process j are all strictly concave.[51]) However, we do not want to assume that the functions $g_{ij}(x_j)$ are strictly concave for all i and j because that would imply that every

[51]) The functions $g_{ij}(x_j)$ are here considered to be strictly concave as functions of the process scale x_j. In nonlinear processes, there may be more than one natural definition of a scale. From one point of view, any strictly monotonic transformation of a scale is itself a scale, since it serves equally well as a parameter in defining the different levels of the process. However, the previous theorems have shown that there is an advantage in choosing the scale so that the functions $g_{ij}(x_j)$ are concave. If this condition does not uniquely specify the choice of a scale, then the scale can be so chosen that all the functions g_{ij} which enter the process can be chosen strictly concave.

It may be worth remarking that the condition that it be possible to choose strictly concave input and output functions is stronger than the condition that the process not be linear. The first condition requires that the process have no linear relations between any input and any output.

commodity enters into every process, either as input or as output. We therefore assume the following less stringent sharpening of $(P - C)$: $(P - SC)$. For each commodity i and process j, the function $g_{ij}(x_j)$ is either strictly concave or identically zero.

We can then state,

Theorem 11. If $(U - SC)$ and $(P - SC)$ hold, and the variables y_i, x_j, and p_i are varied in accordance with the gradient method defined by equations (58—60), all variables converge to their equilibrium values.[52])

E. *The Price-Adjustment Method.*

We will now apply the price-adjustment method presented for constrained maximum problems in part II, section F, to the resource allocation problem. The formulation is obvious; the price-adjustment is the limiting form of the gradient method of (58—60), where the differential equations (58) and (59) are replaced by the conditions that for any given set of prices the helmsman chooses that set of final demands which maximizes the difference between utility and costs and the manager of each process chooses that scale which maximizes profits (55). The totality of these decisions determines supply and demand on each market; the custodian varies price in accordance with (60).

It must be recalled, as already seen in part II, section F, that the instructions to the managers to choose maximizing values for their scales implies that there exists a maximum and that it is unique. The first condition is satisfied when prices are sufficiently close to their equilibrium values but not necessarily everywhere; the second condition requires that the profit function is strictly concave, which is guaranteed if $(P - SC)$ holds. Similarly, the helmsman has a unique maximum in the final demands when prices are close to equilibrium and the utility function is strictly concave. We shall therefore assume that $(P - SC)$ and $(U - SC)$ are both valid.

For any given set of prices, $p_1, \ldots, p_s$, let the helmsman choose $y_1, \ldots, y_n$ so as to maximize $U(y_1, \ldots, y_n) - \sum_{i=1}^{m} p_i y_i$ and the

[52]) Mathematical proofs of the results in this section will be found in K. Arrow and L. Hurwicz, "Gradient Method for Resource Allocation," *op. cit.*

manager of the j^{th} process choose x_j so as to maximize $\pi_j(x_j)$. Denote the choice of helmsman as a function of the p_i's by,

$$\eta_1(p_1, \ldots, p_n), \ldots, \eta_n(p_1, \ldots, p_n).$$

$\eta_i(p_1, \ldots, p_n)$ is the final demand function for the i^{th} desired commodity. The value of x_j chosen by a manager determines the net output, $g_{ij}(x_j)$, of the i^{th} commodity by the j^{th} process; as a function of the prices, it may be denoted by $\gamma_{ij}(p_1, \ldots, p_s)$, the supply (demand if negative) of the i^{th} commodity by the j^{th} process. The aggregate excess supply for the i^{th} commodity, as a function of prices, is then,

$$(61.1)\quad \gamma_i(p_1, \ldots, p_s)=\sum_{j=1}^{m}\gamma_{ij}(p_1, \ldots, p_s)+\xi_i-\eta_i(p_1, \ldots, p_n) \qquad (i = 1, \ldots, n);$$

$$(61.2)\quad \gamma_i(p_1, \ldots, p_s)=\sum_{j=1}^{m}\gamma_{ij}(p_1, \ldots, p_s)+\xi_i \qquad (i = n + 1, \ldots, s).$$

In this notation, the equilibrium conditions for an optimal allocation, as given in Theorem, 9 are,

(62) $\gamma_i(\bar{p}_1, \ldots, \bar{p}_s) \geqq 0$ $(i = 1, \ldots, s)$; for any i such that $\gamma_i(\bar{p}_1, \ldots, \bar{p}_s) > 0$, $\bar{p}_i = 0$,

that is, the equalization of supply and demand with the necessary qualification for free commodities. The equilibrium values of final demands and process scales are determined from the equilibrium prices by the maximization that determined the supply and demand functions. $\bar{y}_i = \gamma_i(\bar{p}_1, \ldots, \bar{p}_n)$, and $\bar{x}_i$ is the profit-maximizing choice which determined the functions γ_{ij}, evaluated for equilibrium prices.

The dynamics of the price-adjustment method are then simply stated:

$$(63)\quad dp_i/dt = \begin{cases} 0 \text{ if } p_i = 0 \text{ and } \gamma_i(p_1, \ldots, p_s) > 0, \\ -\gamma_i(p_1, \ldots, p_s) \text{ otherwise.} \end{cases}$$

Since the price-adjustment method is simply a limiting form of the gradient method, the conditions for its validity are the same. •

Theorem 12. The price-adjustment method defined by (63) converges under assumptions $(U - SC)$ and $(P - SC)$.

F. *Remarks on an Alternative Form of the Price-Adjustment Method.*

The static characterization of an optimum given in section B suggests the following alternative form of the price-adjustment method. The supply and demand functions of the processes, $\gamma_{ij}(p_1, \ldots, p_s)$, are generated in the same way as in part E, but the final demand function, $\bar{\eta}_i(p_1, \ldots, p_n)$, are defined by the maximization of $U(y_1, \ldots, y_n)$ subject to the budget restraint, $\sum_{i=1}^{n} p_i y_i = M$. (The dependence of the final demand $\bar{\eta}_i$ on M is not indicated in the notation, since M is taken as a parameter and remains fixed throughout the adjustment process.) Then the definition (61) of excess supply is altered only by replacing $\eta_i(p_1, \ldots, p_n)$ in (61.1) by $\bar{\eta}_i(p_1, \ldots, p_n)$; let $\bar{\gamma}_i(p_1, \ldots, p_s)$ be the excess supply in the new system. Finally the dynamics are supplied by (62), with $\gamma_i(p_1, \ldots, p_s)$ replaced by $\bar{\gamma}_i(p_1, \ldots, p_s)$.

In this form, the system is a special case of a competitive economy, in which the productive units maximize profits at any given level of prices, the consumers (in this case there is only one) maximize utility at any given level of prices, and prices vary according to the law of supply and demand. This is the model sketched by Walras and formulated more precisely by Samuelson. The conditions under which the equilibrium of such an economy is stable are by no means well known, but a number of sufficient conditions have been established.[53])

IV. MODIFIED MARKET MECHANISMS IN ABSENCE OF DECREASING RETURNS

In this part, we will treat the modifications of the market mechanism which might cope with the problems of resource allocation arising under increasing or constant returns. The treatment must, of course, be much more tentative. The possibility of allocative mechanisms which achieve optimal allocation under increasing returns and still have some measure of decentralization is only beginning to be explored.

[53]) See K. Arrow and L. Hurwicz, "On the Stability of Competitive Equilibrium I," *Econometrica*, Vol. 26 (1958), 522–552; K. Arrow, H. D. Block, and L. Hurwicz, "On the Stability of Competitive Equilibrium II," *ibid*, Vol. 27 (1959), 82-109.

A. *General Remarks.*

As we have seen in several places in part II, the absence of concavity conditions on the functions involved has two consequences for the characterization of maxima (constrained or unconstrained): the first-order conditions do not completely distinguish maxima from other stationary points, and in any case do not in any way distinguish global from merely local maxima.[54] The latter problem cannot be dealt with by any static characterization which is based on derivatives. Correspondingly, no variation of the gradient method, which is based on moving uphill as measured solely by local variations, can be expected to insure arrival at the highest of several peaks; at best, only convergence to a local maximum can be expected.

A simple illustration is provided by an economy with one output, one input, and two processes both operating under increasing returns. Let x_i be the input into the i^{th} process, $f_i(x_i)$ the output of the i^{th} process, and ξ the total amount of input available, so that $x_1 + x_2 = \xi$. We assume $f_i'(x_i) > 0$ $f_i''(x_i) > 0$. We wish to maximize $\psi(x_1) = f_1(x_1) + f_2(\xi - x_1)$. We then have

$$\psi'(x_1) = f_1'(x_1) - f_2'(\xi - x_1),$$
$$\psi''(x_1) = f_1''(x_1) + f_2''(\xi - x_1).$$

We locally maximize $\psi(x_1)$ subject to $0 \leqq x_1 \leqq \xi$. If the maximizer $\hat{x}_1$ satisfies $0 < \hat{x}_1 < 1$, then we must have $\psi'(\hat{x}_1) = 0$, $\psi''(\bar{x}_1) \leqq 0$. But under the assumptions made $\psi''(x_1) > 0$ for all x_1. Hence either $\hat{x}_1 = 0$ (and $\hat{x}_2 = \xi$) or $\hat{x}_1 = \xi$ (and $\hat{x}_2 = 0$); in any case one of the two processes is at zero level.

Now suppose that $f_1'(\xi) > f_2'(0)$ and also $f_2'(\xi) > f_1'(0)$. Then $\psi'(\xi) > 0$ and $\psi'(0) < 0$, so that $x_1 = \xi$ and $x_1 = 0$ are both *local* maxima. (That this can happen is seen from the example $f_i(x_i) = \alpha_i x_i^2$, $\alpha_i > 0$.) If $f_1(\xi) \neq f_2(\xi)$, only one of the two local maxima is a global maximum. (Take $\alpha_1 \neq \alpha_2$ in the preceding example.) Hence we may expect to encounter cases of local maxima which are not global.

We will suppose from now on that we know in a general way

[54]) See part II, sections A.1 and A.5, and the Remark to Theorem 1 in section B

where the global maximum is; the dynamic problem will be that of locating it precisely. Within the framework of gradient methods, nothing better can be expected.

In any case, we will want to seek processes that do not converge to stationary points that are not even local maxima, while if we do start in the neighborhood of a local maximum we wish our approximation methods to converge to it. These, then, are the criteria by which we will judge modifications of the market mechanisms which seek to yield optimal resource allocations under increasing returns. Our mathematical tools are the static characterizations of part II, section C, and their dynamic counterparts, Theorems 6 and 7 of part II, section E.

B. *The Unmodified Lagrangian Conditions.*

For static conditions for an optimal allocation, we start with the Kuhn-Tucker Theorem 1, which has in fact been the basis of discussions of optimal allocation to date. The application of Theorem 1 to the resource allocation problem is entirely parallel to that of Theorem 2 in the case where the concavity conditions necessary for the latter, as given in Theorem 9. Each of the four conditions there has a parallel in the more general case.

The analogue of condition (a) is that,

(64) $\bar{U}_{y_i} \leqq \bar{p}_i \ (i = 1, \ldots, n)$; if $\bar{U}_{y_i} < \bar{p}_i$, then $\bar{y}_i = 0$.

If we retain the concavity conditions $(U - C)$ for the utility function, then the relations (64) are necessary and sufficient for a maximum of the difference, $U(y_1, \ldots, y_n) - \sum_{i=1}^{n} \bar{p}_i y_i$, so that condition (a) of Theorem 9 remains valid.

The analogue of condition (b) is that,

(65) $d\bar{\pi}_j/dx_j \leqq 0$ for $x_j = \bar{x}_j$; if $d\bar{\pi}_j/dx_j < 0$, then $\bar{x}_j = 0$.

In (65), the function $\bar{\pi}_j(x_j)$ is the profit, calculated at equilibrium prices, attached to the j^{th} process, as defined just preceding equations (54).

Finally, conditions (c) and (d) remain valid.

Theorem 13. If $(U - C)$ holds, then the following conditions are necessary for an optimal resource allocation:

(a) the helmsman chooses the final demands $\bar{y}_1, \ldots, \bar{y}_n$ so

as to maximize the difference, $U(y_1, \ldots, y_n) - \sum_{i=1}^{n} \bar{p}_i y_i$, between utility and costs calculated at equilibrium prices;

(b) each process manager chooses a process scale $\bar{x}_j$ such that $d\bar{\pi}_j/dx_j \leqq 0$, but the strict inequality can only hold if $\bar{x}_j = 0$;

(c) for each commodity i ($i = 1, \ldots, s$), the excess supply g_i evaluated at equilibrium final demands and process scales must be non-negative; if positive, the price $\bar{p}_i = 0$.[55])

The difference in conditions for optimal resource allocation between the general case discussed here and that in which concavity assumptions are made about production as well as consumption is solely that between condition (b) of Theorem 13 and condition (b) of Theorem 9. The former requires that at equilibrium prices the marginal profitability of each process be non-positive; it can be negative only if the process is operated at zero scale. The last clause is perhaps more intuitively meaningful in its contrapositive form:

(66) if the marginal profitability is positive at zero output, then the optimal value of the process scale must be positive.

In a general way, the non-negativity clause has been recognized in the literature; it is well known that one of the chief problems in optimal resource allocation under increasing returns is determining which processes are operated at some positive level. Statement (66) supplies a sufficient, though not necessary, condition.

The parallelism between Theorems 9 and 13 suggests a supply-and-demand interpretation. The demand functions of the helmsman are defined precisely as before. For a process manager, however, the supply or demand for each product is defined as a function of all prices by first choosing the process scale x_j so that marginal profitability is zero at the given set of prices (or negative at zero scale) and then finding the corresponding values of $g_{ij}(x_j)$. The equilibrium prices are then defined in the

[55]) It may be remarked that, analogously to Theorem 9, Theorem 13 remains valid when condition (a) is replaced by condition (a′) of Theorem 9′; see part III, section B. Theorem 13 is identical with the propositions of Hotelling, Lange, and Allais, *op. cit.* footnote 43, except for the emphasis on the non-negativity conditions.

usual way as those which equate supply and demand (or lead to an excess supply with zero price).[56]) This interpretation is not as useful as the corresponding one in the case where the concavity conditions on production hold. First, for any given set of prices there is generally more than one value of the process scale which satisfies the stated conditions. Under increasing returns, the marginal profitability can easily increase from a negative value at zero output through zero to positive values. Then both zero and the value of the process scale which makes the marginal profitability zero satisfy the condition. If there is ultimately a phase of diminishing returns, the marginal profitability may again become zero, so there can be three possible values of the process scale. Even more complicated possibilities cannot be excluded. The corresponding supply and demand functions are thus multi-valued and not well defined.

A second difficulty closely related to the first is that the system as a whole will in general have multiple equilibria.[57]) We have already seen that this phenomenon is usual in applications of Theorem 1; see the Remark following. Some of these equilibria will not even be local optima.[58])

If we specify in advance a neighborhood of the optimal allocation and consider only alternatives within it, these two difficulties will in general disappear, and Theorem 13 provides a satisfactory static characterization. The next problem is the determination of a dynamic approximation method which converges to the optimum and is consistent with decentralization. Since the proper aim of a process manager may be to minimize rather than maximize profits, the dynamic system of Theorem 10 cannot apply. The above supply-and-demand interpretation suggests a model in which the dynamic element is simply the

[56]) This formulation is employed by Lange, *Socialism*, pp. 81—82.

[57]) This possibility is admitted by Lange but regarded as exceptional; see *Socialism*, pp. 82, 69—70.

[58]) There is a third problem, of a type which we have not stressed in this paper. If the optimal process scale is such that the marginal profitability is changing from negative values to positive ones, then the position is one of *minimum*, not maximum profits. While this is not a difficulty from the point of view of formal rules of operation, the incentives to the process manager are clearly a good deal less satisfactory than in the concave case where social optima correspond to profit maxima.

adjustment of prices according to supply and demand. The simplest form is the system studied in Theorem 12, with the new interpretation given to the supply and demand functions of processes; in this system price changes at a rate proportional to the negative of excess supply. We will show by example that such a dynamic system can be unstable. However, in section E below we will show that the price-adjustment model will be stable with more complicated rules for price changes.

Let there be two commodities and one process. Commodity 1 is a desired commodity with no initial supplies. Commodity 2 is a primary commodity with an initial supply of 1. The one process has commodity 2 as input and commodity 1 as output. If we use the input as the process scale, x_1, the output is assumed to be $x_1^2/2$, so that the process displays increasing returns. We assume that the single desired commodity 1 is desired at all levels. Then from an ordinalist point of view the utility function can be any strictly increasing function of the final demand y_1. We choose in particular a strictly concave utility function,

$$U(y_1) = (\log y_1)/2. \tag{67}$$

In our notation, the above model can be written

$$g_{11}(x_1) = x_1^2/2, \; g_{21}(x_1) = -x_1, \tag{68}$$

$$\xi_1 = 0, \; \xi_2 = 1. \tag{69}$$

Let us derive the supply and demand functions in the notation of part III, section E. First, the demand function of the helmsman is obtained by maximizing the difference, $U(y_1) - p_1 y_1$, so that

$$\eta_1(p_1) = 1/2p_1. \tag{70}$$

The process manager chooses x_1 to make the marginal profitability zero. Since $\pi_1(x_1) = p_1 g_{11}(x_1) + p_2 g_{21}(x_1) = (p_1 x_1^2/2) - p_2 x_1$, $d\pi_1/dx_1 = p_1 x_1 - p_2 = 0$, so that $x_1 = p_2/p_1$. This choice of process scale generates the following excess supply functions for the two commodities by substitution into (68):

$$\gamma_{11}(p_1, p_2) = p_2^2/2p_1^2, \; \gamma_{21}(p_1, p_2) = -(p_2/p_1). \tag{71}$$

The optimal allocation is obvious by inspection; since the output is always desired and is an increasing function of the

input, we simply set $x_1 = 1$ and therefore $y_1 = 1/2$. If we start in a neighborhood of the optimum, we can disregard non-negativity considerations. The price-adjustment model analogous to that of part III, section E, becomes,

$$(72) \qquad dp_1/dt = -\gamma_1(p_1, p_2) = \eta_1(p_1) - \gamma_{11}(p_1, p_2) - \xi_1 = (1/2p_1) - (p_2/p_1)^2/2,$$

$$(73) \qquad dp_2/dt = -\gamma_2(p_1, p_2) = -\gamma_{21}(p_1, p_2) - \xi_2 = (p_2/p_1) - 1.$$

The equilibrium point of this system can be found by setting the right-hand sides equal to zero, that is, $\gamma_1(\bar{p}_1, \bar{p}_2) = 0$, $\gamma_2(\bar{p}_1, \bar{p}_2) = 0$. From (73), $p_1 = p_2$ at equilibrium; by substitution into (72), we see that $\bar{p}_1 = \bar{p}_2 = 1$. To investigate the local stability of (72—73). we find the matrix of partial derivatives of the right-hand sides of (72—73) with respect to p_1 and p_2, evaluated at the equilibrium point; the condition for stability is that the real parts of the characteristic roots of this matrix be negative.[59]) Straightforward calculation shows that the matrix is,

$$\begin{pmatrix} 1/2 & -1 \\ -1 & 1 \end{pmatrix}.$$

The characteristic equation is a quadratic with roots $[3 \pm \sqrt{17}]/4$, so that one characteristic root is positive, and the system (72—73) is unstable.[60])

C. *A Digression on Marginal-Cost Pricing.*

Suppose that the j^{th} process has a single output, say com-

[59]) See Samuelson, *op. cit.*, pp. 270—74.

[60]) There is another gradient method which corresponds to Theorem 13 (or, more generally, to Theorem 1) and which does converge in general to a local maximum. It is defined by moving in a direction which is as close to the gradient of the unconstrained maximization as possible compatible with satisfying the constraints at all times. In the resource allocation model, this would require that the excess supply for each commodity would have to be zero (apart from free goods) at all points in the approximation process, not merely at equilibrium. There seems no way of satisfying this requirement in a decentralized fashion. For this gradient process, see G. E. Forsythe, "Computing Constrained Minima with Lagrange Multipliers," *Journal of the Society for Industrial and Applied Mathematics*, Vol. 3 (1955), 173—78; Arrow and Solow, *op. cit.*, sec. 3.

modity 1, and that the scale of the process is taken to be that output. Then $g_{1j}(x_j) = x_j$, and $-g_{ij}(x_j)$ is the input of commodity i needed to produce one unit of commodity 1 ($i = 2, \ldots, s$). Clearly,

$$(74) \qquad d\bar{\pi}_j/dx_j = \bar{p}_1 - \sum_{i=2}^{s} \bar{p}_i[-g'_{ij}(x_j)].$$

Since $-g'_{ij}(x_j)$ is the increase in input i per unit increase in output, (74) may be interpreted in familiar fashion; the marginal profitability equals price less marginal cost. Condition (b) then says that at the optimum price equals marginal cost, except that for a process not operated at all price may be less than marginal cost.[61])

We have thus the familiar rule of marginal cost pricing as a condition for optimal allocation under increasing as well as diminishing returns, that is, that for a process operated at some positive level the output should be chosen so that the marginal cost equals the price. Under diminishing returns, the marginal cost curve will be rising, and the rule then chooses the point of maximum profit. Under increasing returns, however, the marginal cost may be falling, and the (socially) optimal output will yield minimum profit.

Apart from the non-negativity condition, then, the usual marginal-cost-pricing condition is correct when applied, as in our model, to individual processes. But the condition is usually applied to firms, where a firm, in our formulation, is a single decision-making unit having control over several processes. In this form, it is *not* correct, contrary to prevailing opinion, that the optimal allocation corresponds in general to an equality of price and marginal cost, as the latter term is usually defined.

To see this, recall that the cost function is defined in economic

[61]) Actually, this interpretation is not dependent upon identifying the scale of process with the output. We can always write,

$$d\bar{\pi}_j/dx_j = g'_{1j}(x_j)\ \{\bar{p}_1 - \sum_{i=2}^{s} [\bar{p}_i - g_{ij}(x_j)/g'_{1j}(x_j)]\}.$$

The ratio, $-g'_{ij}(x_j)/g'_{1j}(x_j)$, is clearly the marginal increase in input i for a unit increase in output; hence the term in braces is the difference between price and marginal cost. Since the factor, $g_{1j}(x_j)$, is necessarily positive, the statement in the text is always equivalent to condition (b).

literature as the *minimum* cost of producing any given output, prices being taken as given.[62]) Marginal cost is then simply the derivative of the cost function with respect to output.

The usual rule of equaling marginal cost to price, then, implies that the output of the firm is produced at minimum cost. Despite the reasonable sound of this statement, it is not, in general, a correct rule for optimal resource allocation. Consider for example a firm which has two processes with the same output but different inputs, each process showing increasing returns. Let $f_1(y_1)$ be the amount of input 1 required to produce amount y_1 of the output by process 1, $f_2(y_2)$ the amount of input 2 required to produce amount y_2 of the output by process 2. Let ξ_1 and ξ_2 be the amounts of inputs 1 and 2 initially available. We assume increasing returns, so that $f_1''(y_1) < 0$, $f_2''(y_2) < 0$. The rule of minimizing cost at any given output requires the form to choose y_1 and y_2 so as to minimize $p_1f_1(y_1) + p_2f_2(y_2)$ subject to the constraint, $y_1 + y_2 = y$, where p_1 and p_2 are the prices of inputs 1 and 2 respectively. This is equivalent to minimizing,

$$\psi(y_1) = p_1f_1(y_1) + p_2f_2(y - y_1),$$

with respect to y_1 subject to the inequality constraints $0 \leqq y_1 \leqq y$. By differentiation, $\psi''(y_1) < 0$ throughout the interval, so that the minimum must occur at either $y_1 = 0$ or $y_1 = y$, i.e., $y_2 = 0$. Thus, no matter what values are assigned for p_1, p_2, and y, one of the two inequalities,

$$f_1(y_1) < \xi_1, \; f_2(y_2) < \xi_2,$$

must hold.

On the other hand socially optimal allocation requires maximizing total output $y_1 + y_2$ subject to the constraints,

[62]) For representative texts, see G. J. Stigler, *The Theory of Price,* (rev. ed.; New York: Macmillan, 1952) pp. 127—29; K. E. Boulding, *Economic Analysis,* (3rd ed.; New York: Harpers, 1955) ch. 34. This definition of cost is clearly the correct one in the theory of the firm, whether under competition or under monopoly; maximization of profit requires maximizing the difference between revenue and cost as defined in the text, or, equivalently, the equating of marginal revenue and marginal cost.

Writers who have dealt with optimal allocation under increasing returns have usually not been careful to define marginal costs; however, Lange explicitly presents the definition given in the text (pp. 116—17).

$f_1(y_1) \leqq \xi_1$, $f_2(y_2) \leqq \xi_2$: It is obvious that the maximum is attained by choosing y_1 and y_2 so that,

$$f_1(y_1) = \xi_1,\ f_2(y_2) = \xi_2.$$

Therefore it is impossible that the rule of cost minimization lead in this case to an optimal allocation of resources. The minimum cost solution is not socially efficient because it leaves unemployed resources which have positive usefulness, no matter how factor prices are chosen.

D. *Optimal Allocation Through Imperfect Competition.*

To deal with the drawbacks of Theorem 1 as a tool for determining optimal resource allocation under increasing returns, we will turn to Theorems 3 and 4 which yield local saddle-point characterizations of the optimal allocation. Unlike Theorem 1, these conditions are satisfied only by local maxima, not by other extrema.

First, consider the application of Theorem 3 to the resource application problem. The concavified Lagrangian with transformed constraints becomes,

(75)
$$L^*(y_1, \ldots, y_n, x_1, \ldots, x_m; p_1, \ldots, p_s/\eta) = U(y_1, \ldots, y_n) + \sum_{i=1}^{s} p_i[1 - (1 - g_i)^{1+\eta_i}],$$

where the excess supply $g_i = g_i(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is defined by (46) and (47).

The optimal allocation is characterized as a local saddle-point. Since L^* is linear in the p_i's, with respect to which there is a minimum, we have,

(76) $1 - (1 - \bar{g}_i)^{1+\eta_i} \geqq 0$; for any i for which the strict inequality holds, $\bar{p}_i = 0$.

Here, $\bar{g}_i$ is the excess supply at equilibrium, that is, $g_i(\bar{x}_1, \ldots, \bar{x}_m, \bar{y}_1, \ldots, \bar{y}_n)$. However, it is easy to see that the expression,

$$1 - (1 - g_i)^{1+\eta_i},$$

is positive, zero, or negative according as g_i is positive, zero, or negative; indeed it was originally chosen to have this property

(see remarks preceding (37) in part II, section C). Hence (76) is equivalent to,

$$(77) \qquad \bar{g}_i \geqq 0; \text{ if } \bar{g}_i > 0, \text{ then } \bar{p}_i = 0,$$

as was to be expected.

Let us combine the maximization part of the saddle-point criterion with (77).

Theorem 14. Under certain regularity conditions, a necessary and sufficient condition for a locally optimal allocation is that,

(a) $\bar{y}_1, \ldots, \bar{y}_n, \bar{x}_1, \ldots, \bar{x}_m$ maximize the expression, $U(y_1, \ldots, y_n) + \sum_{i=1}^{s} \bar{p}_i (1 - [1 - g_i(x_1, \ldots, x_m, y_1, \ldots, y_n)]^{1+\eta_i})$;

(b) $g_i(\bar{x}_1, \ldots, \bar{x}_m, \bar{y}_1, \ldots, \bar{y}_n) \geqq 0$; if $g_i(\bar{x}_1, \ldots, \bar{x}_m, \bar{y}_1, \ldots, \bar{y}_n) > 0$, then $\bar{p}_i = 0$, where the η_i's have been chosen sufficiently large.

Condition (a) can be given an institutional interpretation. Suppose the helmsman takes over the functions of the process managers in addition to his own, so that he chooses the x_j's as well as the y_i's. At the same time, suppose that the custodian for commodity i purchases any amount u_i available for an amount, $\bar{p}_i[1 - (1 - u_i)^{1+\eta_i}]$. This function is a total revenue function in the i^{th} market. The helmsman then maximizes the sum of his utility and the revenue he obtains by selling the excess supply (the "revenue" would, of course, be negative if there were an excess demand). The helmsman on this interpretation is dealing with a series of markets on which he is an monopolist or monopsonist.

In this formulation, the p_i's do not play the role of competitive prices, but they are parameters which affect the location, though not the general shape, of the revenue functions. We will therefore refer to them as *revenue parameters*. Condition (b) insures that at equilibrium the revenue parameters are chosen so that the helmsman will have zero excess supply for free goods (when $\bar{p}_i = 0$, the revenue is identically zero).

So far the institutional interpretation has been highly centralized. All that has been done is to rephrase the constrained

maximization problem as two problems, one an unconstrained maximization and one a feasibility condition. However, any unconstrained maximization has a certain element of decentralization implicit in it. If $\bar{z}_1, \ldots, \bar{z}_p$ maximize a function $f(z_1, \ldots, z_p)$, then it is true that $\bar{z}_1$ maximizes the function $f(z_1, \bar{z}_2, \ldots, \bar{z}_p)$ as a function of the single variable z_1, and similarly with the other z_i's. In the case of Theorem 9, there was an extreme simplification in the unconstrained maximization (given the $\bar{p}_i$'s), since the function to be maximized was a sum of functions each involving a different variable. The maximization in Theorem 14 is not so simple, but along the lines just suggested there is a degree of decentralization which, as we shall see, becomes more pronounced in the dynamic form.

Let,

$$h_{ij} = g_i - g_{ij} \quad (i = 1, \ldots, s;\ j = 1, \ldots, m); \tag{78}$$

$$h_i = g_i + y_i \quad (i = 1, \ldots, n). \tag{79}$$

Reference to (46—47) shows that h_{ij} is independent of x_j, though it depends on the scales of all processes other than the j^{th} and, in the case of desired commodities, on y_i. Similarly, h_i depends on the process scales but is independent of the y_i's. If, following the argument above, we maximize with respect to x_j, taking the y_i's and all process scales other than the j^{th} as given at their equilibrium values, we have that,

$$\bar{x}_j \text{ maximizes } \sum_{i=1}^{s} \bar{p}_i \{1 - [1 - \bar{h}_{ij} - g_{ij}(x_j)]^{1+\eta_i}\} \tag{80}$$

the term $U(\bar{y}_1, \ldots, \bar{y}_n)$ being omitted since it is a constant with respect to x_j. Here as before a bar over a symbol denotes evaluation at equilibrium. Similarly,

$$\bar{y}_1, \ldots, \bar{y}_n \text{ maximizes } U(y_1, \ldots, y_n) + \sum_{i=1}^{s} \bar{p}_i [1 - (1 - \bar{h}_i + y_i)^{1+\eta_i}]. \tag{81}$$

Equations (80) and (81) suggest a degree of decentralization. Each process manager chooses an x_j so as to maximize (80), the helmsman chooses final demands so as to maximize (81). Each is now operating on imperfectly competitive markets. Each now needs somewhat more information from the outside than in the

concave case. The manager of the j^{th} process needs to know not only the $\bar{p}_i$'s but also the $\bar{h}_{ij}$'s $(i = 1, \ldots, s)$; the helmsman needs to know both $\bar{p}_i$ and $\bar{h}_i$ $(i = 1, \ldots, n)$.

The dynamic counterpart of Theorem 14 is derived from Theorem 6. If we apply the theorem to the resource allocation, we find by straightforward differentiation that,

$$L^*_{y_i} = U_{y_i} - (1 + \eta_i)p_i(1 - g_i)^{\eta_i}, \tag{82}$$

$$L^*_{x_j} = \sum_{i=1}^{s} (1 + \eta_i)p_i(1 - g_i)^{\eta_i}g'_{ij}. \tag{83}$$

The gradient method based on these derivatives requires the helmsman and the process managers to find the unconstrained maxima of (81) and (80) respectively, with equilibrium replaced by current values. The information required is greater than in the concave case of Theorems 10 and 11, but not too much greater. The helmsman and each process manager must now know the current values of both the revenue parameter and the excess supply for every market, in addition to his utility function or technology, respectively.

The behavior of the revenue parameters is a straightforward dynamic analogue of (76).

Theorem 15. Under certain regularity conditions and for η_i's sufficiently large, the system of differential equations,

(a) $dy_i/dt = \begin{cases} 0 \text{ if } L^*_{y_i} < 0 \text{ and } y_i = 0, \\ L^*_{y_i} \text{ otherwise,} \end{cases}$

(b) $dx_j/dt = \begin{cases} 0 \text{ if } L^*_{x_j} < 0 \text{ and } x_j = 0, \\ L^*_{x_j} \text{ otherwise;} \end{cases}$

(c) $dp_i/dt = \begin{cases} 0 \text{ if } [1 - g_i(x_1, \ldots, x_m, y_1, \ldots, y_n)]^{1+\eta_i} < 1 \\ \text{and } p_i = 0, \\ [1 - g_i(x_1, \ldots, x_m, y_1, \ldots, y_n)]^{1+\eta_i} - 1 \text{ otherwise;} \end{cases}$

converges to a local optimum of allocation. Here $L^*_{y_i}$ and $L^*_{x_j}$ are defined by (82) and (83), respectively.

Theorem 15 thus supplies a dynamic system which converges locally to an optimum and which makes very limited informational demands upon its participants.

Theorem 4 leads to a static characterization of an optimal

allocation similar to that of Theorem 14. The new feature is the existence of a set of variables, the w_i's, which, in equilibrium, represent the permissible excess supply. If we refer back to the reasoning leading to Theorem 4, we see that the w_i's were introduced to convert inequality constraints into equality constraints. At equilibrium, then equality must hold in the constraints $g_i(x_1, \ldots, x_m, y_1, \ldots, y_n) - w_i = 0$.

$$\bar{g}_i = \bar{w}_i; \text{ if } \bar{g}_i > 0, \text{ then } \bar{p}_i = 0. \tag{84}$$

The second part of (84) is intuitively obvious and can be proved rigorously.

In the saddle-point characterization supplied by Theorem 4, the concavified Lagrangian with quadratic modification has as maximizing variables the w_i's as well as the x_j's and the y_i's. However, if we follow the argument given above about the decentralization implicit in any maximization we may say that the helmsman and the process managers maximize the Lagrangian with respect to the variables under their control, taking as given both the $\bar{p}_i$'s and the $\bar{w}_i$'s. We may refer to the w_i's as the *disposal parameters*. Then, given both the revenue and the disposal parameters, the helmsman and the process managers have, just as in Theorem 14, to maximize a nonlinear function of the inputs and outputs. Again they can be thought of as seeking to maximize profits and utilities by activities which include trading on imperfect markets.

Theorem 16. Under certain regularity conditions and for λ sufficiently large, a necessary and sufficient condition for a locally optimal allocation is that,

(a) $\bar{y}_1, \ldots, \bar{y}_n, \bar{x}_1, \ldots, \bar{x}_m$ maximize the expression,

$$U(y_1, \ldots, y_n) + \sum_{i=1}^{s} [\bar{p}_i(g_i - \bar{w}_i) - \lambda(g_i - \bar{w}_i)^2];$$

(b) $\bar{g}_i = \bar{w}_i \geqq 0$; if $\bar{w}_i > 0$, then $\bar{p}_i = 0$.

The dynamic counterpart of Theorem 16 derives from application of Theorem 6 to the resource allocation problem. The differentiations involved are straightforward.

Theorem 17. Under certain regularity conditions and for λ sufficiently large, the system of differential equations,

(a) $dy_i/dt = \begin{cases} 0 \text{ if } L^\dagger_{y_i} < 0 \text{ and } y_i = 0, \\ L^\dagger_{y_i} \text{ otherwise,} \end{cases}$

(b) $dx_j/dt = \begin{cases} 0 \text{ if } L^\dagger_{x_j} < 0 \text{ and } x_j = 0, \\ L^\dagger_{x_j} \text{ otherwise,} \end{cases}$

(c) $dw_i/dt = \begin{cases} 0 \text{ if } p_i - 2\lambda(g_i - w_i) > 0 \text{ and } w_i = 0, \\ -p_i + 2\lambda(g_i - w_i) \text{ otherwise,} \end{cases}$

(d) $dp_i/dt = \begin{cases} 0 \text{ if } g_i > w_i \text{ and } p_i = 0, \\ w_i - g_i \text{ otherwise,} \end{cases}$

converges to a local optimum of allocation. In (a) and (b), respectively,

(e) $L^\dagger_{y_i} = U_{y_i} - p_i + 2\lambda(g_i - w_i)$,

(f) $L_{x_j} = \sum_{i=1}^{s} [p_i - 2\lambda(g_i - w_i)]g'_{ij}$.

The adjustment formulas for the individual participants are comparatively simple. The helmsman and each process manager have to know, in addition to their own utility function or technology, the revenue parameter, p_i, the disposal parameter, w_i, and the excess supply, g_i, for each commodity. The custodian for commodity i has to know g_i and he can then determine both the revenue and the disposal parameters from the pair of differential equations (c) and (d).

E. *Optimal Allocation Through Nonlinear Price Adjustment.*

If we examine (82) and (83), we see that there is a common expression in all of them, for which we may introduce a symbol,

$$q_i = (1 + \eta_i)p_i(1 - g_i)^{\eta_i}. \tag{85}$$

If we now refer to (a) and (b) of Theorem 15, we see that the helmsman or the process managers can respectively be thought of as seeking to maximize the difference between utility and costs or profits, if the q_i's are now regarded as prices. These prices are now determined by a fairly complicated adjustment pattern; first an auxiliary variable p_i is determined by equation (c) of Theorem 15 and then q_i is calculated from (85). However, the informational requirements are low, since the custodian, who

is charged with these operations, need still know at any one time only the excess supply g_i.

Theorem 18. Under certain regularity conditions and for η_i's sufficiently large, the dynamic system,

(a) $dy_i/dt = \begin{cases} 0 \text{ if } U_{y_i} < q_i \text{ and } y_i = 0, \\ U_{y_i} - q_i \text{ otherwise,} \end{cases}$

(b) $dx_j/dt = \begin{cases} 0 \text{ if } d\tilde{\pi}_j/dx_j < 0 \text{ and } x_j = 0, \\ d\tilde{\pi}_j/dx_j \text{ otherwise,} \end{cases}$

(c) $q_i = (1 + \eta_i)p_i(1 - g_i)^{\eta_i}$,

(d) $dp_i/dt = \begin{cases} 0 \text{ if } g_i > 0 \text{ and } p_i = 0, \\ (1 - g_i)^{1+\eta_i} - 1 \text{ otherwise,} \end{cases}$

converges to a local optimum of allocation. In (b),

(e) $\tilde{\pi}_j(x_j) = \sum_{i=1}^{s} q_i g_{ij}(x_j)$.

In Theorem 18, the informational requirements on each participant are no different than in the concave case of Theorem 10. The only elements of centralization are that a prior decision must be made on the constants η_i and that a global optimum can only be assured if the initial approximation is sufficiently close.

A limiting case of some interest arises if the adjustment speed of the helmsman and the process managers are increased indefinitely. In that case, for the helmsman we must have $U_{y_i} = q_i$ unless $y_i = 0$ and similarly $d\tilde{\pi}_j/dx_j = 0$ unless $x_j = 0$.[63] Then we return to the supply-and-demand model briefly discussed in section B. But the price-adjustment rules of (c) and (d) of Theorem 18 now insure stability.

Theorem 19. Theorem 18 remains valid if (a) and (b), are replaced by,

[63]) It should be made clear that in this process x_j is *not* necessarily chosen to maximize process profits, taking the q_i's as given prices. The chosen x_j does maximize the expression (80), with $\bar{p}_i$ and $\bar{h}_{ij}$ being replaced by their current values, p_i and h_{ij}; with the notation of (85), the statement in the text follows. Thus, as Lange correctly noted, the rule that marginal profitability be zero replaces the rule of profit maximization for increasing returns; however, the above analysis shows that the price-adjustment rule which leads to a stable allocative system is not a simple one.

(a′) $U_{y_i} \leqq q_i$; if $U_{y_i} < q_i$, then $y_i = 0$;
(b′) $d\tilde{\pi}_j/dx_j \leqq 0$; if $d\tilde{\pi}_j/dx_j < 0$, then $x_j = 0$.

F. *Optimal Allocation Through Price Speculation.*

An alternative institutional interpretation can be given of Theorem 17, in a manner somewhat similar to that which led from Theorem 15 to Theorem 18. If we look at (e) and (f) of Theorem 17, we find it natural to introduce the definition,

$$p_i^f = p_i + 2\lambda(w_i - g_i). \tag{86}$$

If we now take account of (d) of Theorem 17, we see that, unless both $g_i > w_i$ and $p_i = 0$,

$$p_i^f = p_i + 2\lambda(dp_i/dt), \tag{87}$$

that is, p_i^f can be considered as an expectation of the price of commodity i in the near future formed by extrapolation of current rates of change.[64]) We will now argue that (87) can be made to hold without the exception noted above. As noted in the Remark following Theorem 4, the saddle-point properties of the concavified Lagrangian with quadratic modification hold whether the p_i's are considered to be non-negative or unrestricted as to sign. Relation (d) of Theorem 17 is the appropriate form if the p_i's are restricted to be non-negative. If they are taken to be unrestricted as to sign, then only the second line is relevant; Theorem 17 remains true with this alteration. Then (87) is true in general.

Theorem 20. Under certain regularity conditions and for λ sufficiently large, the system of differential equations,

(a) $dy_i/dt = \begin{cases} 0 \text{ if } U_{y_i} < p_i^f \text{ and } y_i = 0, \\ U_{y_i} - p_i^f \text{ otherwise,} \end{cases}$

[64]) *Extrapolative expectations* of the type of equation (87) have been studied in connection with inventories by L. Metzler, "The Nature and Stability of Inventory Cycles," *Review of Economic Statistics*, XXIII (1941), 113—29, and in connection with inflationary price movements by A. C. Enthoven, "Monetary Disequilibria and the Dynamics of Inflation," *Economic Journal*, Vol. 66 (1956), 256—70; see also, A. C. Enthoven and K. Arrow, "A Theorem on Expectations and the Stability of Equilibrium," *Econometrica*, Vol. 24 (1956) 288—293. More detailed study of the system studied in this section will be found in Arrow and Solow, *op. cit.*, sec. 4.

(b) $dx_j/dt = \begin{cases} 0 \text{ if } d\pi_j^f/dx_j < 0 \text{ and } x_j = 0, \\ d\pi_j^f/dx_j \text{ otherwise,} \end{cases}$

(c) $dw_i/dt = \begin{cases} 0 \text{ if } p_i^f > 0 \text{ and } w_i = 0, \\ -p_i^f \text{ otherwise,} \end{cases}$

(d) $dp_i/dt = w_i - g_i,$

(e) $p_i^f = p_i + 2\lambda(dp_i/dt),$

converges to a local optimum of allocation. In (b),

(f) $\pi_j^f(x_j) = \sum_{i=1}^{s} p_i^f g_{ij}(x_j).$

Relations (a) and (b) show that the helmsman and the process managers are moving in the direction of maximum benefits computed at expected prices, rather than current ones. Equations (c—e) can be given various institutional interpretations; one is that, taken together, they instruct the custodian how to determine the expected price to be announced to the other participants. It is required that the expected price be sufficiently sensitive to the current rate of change.[65])

G. *The Linear Case.*

We return finally to the case which started the entire investigation, that where the utility function and all the processes are linear, that is, the case of linear programming. In this case, as we have seen in part II, section D, the gradient method will normally lead to endless oscillations. Since stability has been shown to be related to the strict concavity of the Lagrangian, it is natural to try one of the concavification methods introduced in part II, section C. Since the problem is linear, the Lagrangian is, so to speak, on the borderline of strict concavity, and it might be expected that any modification of the Lagrangian which increases its concavity would suffice. This expectation is correct, at least as far as the method of concavification by transformed constraints.

[65]) The system of Theorem 20 and its interpretation in terms of price expectations is similar but not identical to the proposal of T. Kose, "Solutions of Saddle Value Problems by Differential Equations," *Econometrica*, Vol. 24 (1956), 59—70.

Theorem 21. If $U(y_1, \ldots, y_n)$ and the process functions $g_{ij}(x_j)$ are all linear, then the system of differential equations of Theorem 15, with any $\eta_i > 0$ converges to a global optimum of allocation. Equivalently, the dynamic systems of Theorems 18 and 19 converge to a global optimum under the same condition.[66]

The method of Theorem 21 is by no means the only gradient method for the solution of linear programming problems. Because of their equivalence to zero-sum two-person games,[67] the modified gradient methods developed by Brown and von Neuman for solving the latter can be applied to the former.[68] Of course, there are other methods, of which the best known is the simplex method, which are essentially different from the gradient methods. Although there has been as yet insufficient computational experience, it is very likely that the simplex method is a superior method for computation to any variety of the gradient method. But from the viewpoint of the present study, it lacks the very important virtue of decentralization.

[66] For a proof of Theorem 21, see Arrow and Hurwicz, "Gradient Method for Resource Allocation," *op. cit.*, Theorem 3. An illustration of this method with remarks on the problems of machine calculation is found in T. Marschak, "An Example of a Modified Gradient Method for Linear Programming," ch. 9 in Arrow, Hurwicz, and Uzawa, *op. cit.*

[67] See footnote 37.

[68] See G. W. Brown and J. von Neumann, "Solutions of Games by Differential Equations," in H. W. Kuhn and A. W. Tucker, eds., *Contributions to the Theory of Games I*, Annals of Mathematics Study No. 29 (Princeton, New Jersey: Princeton University Press, 1950), pp. 73—80. For the application to linear programming, see M. Fukuoka, "A note on Convergence in Linear Programming Problems," Cowles Commission Discussion Paper: Economics No. 2108, July 14, 1954.

PART II

On the Theory of Utility and Demand

GERHARD TINTNER

External Economies in Consumption[1].

The importance of external economies in production is well recognized. Recent literature has treated the theory of external economies in consumption.[2] This may have some importance for welfare economics.[3] It is one way of getting away from the unrealistic assumption of a society consisting of atomistic individuals, who do not influence each other. Such a procedure has the advantage of simplicity, but as Aristotle remarked, man is a political animal (Politics, I.29 1252b).

We assume a society consisting of K individuals. Let $u^{(k)}$ be the utility function or utility index of the individual k $(k = 1, 2 \ldots K)$. Let x_{kn} be the amount of the commodity or service n consumed by the individual k $(n = 1, 2 \ldots N)$. We assume that the money income M_k of the individual is given, as well as the prices p_n $(n = 1, 2 \ldots N)$ of all commodities and services. We denote by:

$$(1) \qquad X_n = \sum_{k=1}^{K} x_{kn} \quad (n = 1, 2 \ldots N)$$

the total amount of commodity or service n demanded. We neglect the possibility of coalitions and other complications arising in the theory of games.

1) I am much obliged to Dr. R. L. Basman (Hanford, Wash.) and to M. Wimbée (Paris, France) for advice and criticism of this paper. Journal Paper No. J-3766 of the Iowa Agricultural and Home Economics Experiment Station, Ames Iowa. Project No. 1200.

2) J. S. Duesenberry, *Income, Savings and the Theory of Consumer's Behavior.* (Cambridge, Mass.: Harvard University Press, 1949), H. G. Johnson, "The Effects of Income Redistribution on Aggregate Consumption with Interdependence of Consumer's Preferences," *Economica*, vol. 19 (1952), 131 ff. J. S. Prais and H. S. Houttakker, *The Analysis of Family Budgets* (Cambridge: Cambridge University Press, 1955).

3) G. Tintner, "A Note on Welfare Economics," *Econometrica*, vol. 14 (1946), 69 ff.

Following Allen,[4]) we denote by:

$$Ef/Ez = (\partial f/\partial z)(z/f) \tag{2}$$

the elasticity of a function f with respect to a variable z.

We present first the *classical* atomic theory.[5]) Let

$$u^{(k)} = u^{(k)}(x_{k1}, x_{k2}, \ldots x_{kN}) \tag{3}$$

be the utility function of the individual k, which depends only upon the quantities of commodities and services possessed by himself. The function (3) has to be maximized under the budget restriction:

$$\sum_{n=1}^{N} x_{kn} p_n = M_k. \tag{4}$$

Now we define:

$$\partial u^{(k)}/\partial x_{kr} = u^{(k)}_{kr};\ \partial^2 u^{(k)}/\partial x_{kr}\partial x_{ks} = u^{(k)}_{kr,\,ks} \tag{5}$$

and the matrix with $N+1$ rows and columns:

$$U^{(k)} = \begin{bmatrix} 0 & u^{(k)}_{kr} \\ u^{(k)}_{ks} & u^{(k)}_{kr,\,ks} \end{bmatrix} \qquad (r,\ s = 1, 2 \ldots N). \tag{6}$$

Let $U^{(k)}_{kr}$ be the cofactor of $u^{(k)}_{kr}$ in the inverse matrix $U^{(k)-1}$ and $U^{(k)}_{kr,\,ks}$ the cofactor of $u^{(k)}_{kr,\,ks}$ in the same matrix. Then we have the well known results:

$$\partial x_{ks}/\partial M_k = L_k U^{(k)}_{ks} \tag{7}$$

$$\partial x_{ks}/\partial p_r = -L_k U^{(k)}_{ks} x_{kr} + L_k U^{(k)}_{kr,\,ks} = -x_{kr}(\partial x_{ks}/\partial M_k) + L_k U_{kr,ks}. \tag{8}$$

In these formulae L_k is a Lagrange multiplier, the marginal utility of money for individual k. We have for superior commodities:

$$\partial x_{ks}/\partial M_k > 0 \tag{10}$$

and for inferior commodities:

$$\partial x_{ks}/\partial M_k < 0. \tag{11}$$

$L_k U^{(k)}_{kr,\,ks}$ is the substitution term. If

$$U^{(k)}_{kr,\,ks} < 0 \tag{12}$$

[4]) R. G. D. Allen, *Mathematical Analysis for Economists* (London: Macmillan, 1949), pp. 251 ff.

[5]) J. R. Hicks, *Value and Capital* (2nd ed., Oxford: Clarendon Press, 1956). P. A. Samuelson, *Foundations of Economic Analysis*, (Cambridge, Mass.: Harvard University Press, 1947, H. Hotelling, "Demand Functions with Limited Budgets," *Econometrica*, vol. 3 (1935) 66 ff.

then the commodities r and s are substitutes. If

$$U^{(k)}_{kr,\,ks} > 0 \tag{13}$$

then r and s are complementary commodities or services.

We see from formula (8) that we may have $\partial x_{ks}/\partial p_s > 0$, in spite of the fact that $U^{(k)}_{ks,\,ks} < 0$. This is the Giffen phenomenon which occurs for inferior goods.

Now we define:

$$m_{kr} = p_r x_{kr}/M_k \tag{14}$$

the proportion of the expenditure of individual k on commodity r in relation to his total expenditure or income. Also the partial elasticity of substitution:

$$\sigma^{(k)}_{kr,\,ks} = (\sum_{n=1}^{N} x_{kn} u^{(k)}_{kn}) U^{(k)}_{kr,\,ks}/(x_{kr} \cdot x_{ks}) \tag{15}$$

The elasticity with respect to price is now given by:

$$Ex_{ks}/Ep_r = m_{kr}(\sigma_{kr,\,ks} - Ex_{ks}/EM_k). \tag{16}$$

For the total demand in the market for commodity s we have:

$$\partial X_s/\partial M_k = \partial x_{ks}/\partial M_k \tag{17}$$

$$\partial X_s/\partial p_r = \sum_{k=1}^{K} \partial x_{ks}/\partial p_r \tag{18}$$

and for the elasticities:

$$EX_s/EM_k = Ex_{ks}/EM_k \tag{19}$$

$$EX_s/Ep_r = \sum_{k=1}^{K} (Ex_{ks}/Ep_r)(x_{ks}/X_s). \tag{20}$$

The price elasticity of demand for the total demand in the market is the weighted sum of the individual demand elasticities, the weights are the proportions x_{ks}/x_s of the quantity demanded by individual k in relation to the total demand for the commodity in question.

Now we assume that the utility index depends on *all the incomes* in the society:

$$v^{(k)} = v^{(k)}(x_{k1}, x_{k2} \ldots x_{kN};\ M_1,\ M_2 \ldots M_K) \tag{21}$$

This is perhaps a somewhat realistic assumption, since the tastes of individuals depend upon the social class of the individuals

and social standing is to some extent depending upon income, at least in America.[6])

We define:

$$\partial v^{(k)}/\partial M_t = v_{(t)}^{(k)}. \tag{22}$$

Now we have:

$$\partial x_{ks}/\partial M_t = L_k U_{ks}^{(k)} \cdot \delta_{kt} - \sum_{n=1}^{N} U_{ks,\,kn}^{(k)} v_{kn(t)}^{(k)}. \tag{23}$$

In this formula δ_{kt} denotes the Kronecker delta $\delta_{kt} = 1$ if $k = t$, $\delta_{kt} = 0$ otherwise.

The income elasticity appears now as:

$$Ex_{ks}/EM_t = \delta_{kt}(Ex_{ks}/EM_k)_0\,(M_t/M_k) - \sum_{n=1}^{N} (Ev_{kn}^{(k)}/EM_t)\, m_{kn}\, \sigma_{ks,\,kn}^{(k)}. \tag{24}$$

In this formula $(Ex_{ks}/EM_k)_0$ is the income elasticity as defined in formula (11) for the atomic case. Let us put $t = k$, i.e. we consider the elasticity with respect to the income of the individual k in question:

$$Ex_{ks}/EM_k = (Ex_{ks}/EM_k)_0 - \sum_{n=1}^{N} (Ev_{kn}^{(k)}/EM_k)\sigma_{ks,\,kn}^{(k)}. \tag{25}$$

We see from this formula, that income elasticities and elasticities of substitution are no longer independent. Because of the dependence of the utility function on all incomes, we may have in (25) $Ex_{ks}/EM_k < 0$ even if the atomistic income elasticity $(Ex_{ks}/EM_k)_0$ is positive, i.e. we have a superior good in the classical sense. This is due to the sum in formula (25), which may be large and positive.

Similarly we have for the total demand:

$$EX_s/EM_t = (Ex_{ks}/EM_t)_0 - \sum_{k=1}^{K}\sum_{n=1}^{N} (Ev_{kn}^{(k)}/EM_t)\sigma_{ks,\,kn}^{(k)}. \tag{26}$$

Now we consider the most general case in which the utility of each individual depends upon the *consumption of all individuals* in the given society. We may think of the Veblen effect and similar phenomena.[7])

[6]) T. Parsons and N. J. Smelser, *Economy and Society* (London: Routledge, 1956).

[7]) T. Veblen, *Theory of the Leisure Class* (New York: Modern Library, 1945).

$$(27)\quad w^{(k)} = w^{(k)}(x_{11}, x_{12} \ldots x_{1N}, x_{21}, x_{22}, \ldots x_{2N} \ldots x_{K1}, x_{K2} \ldots x_{KN}).$$

We define a compound matrix, i.e. a matrix consisting of matrices:

$$(28)\quad W = [A^{(ij)}], \quad (i, j = 1, 2 \ldots K),$$

where the individual matrices are defined as follows:

$$(29)\quad A^{(ij)} = \begin{bmatrix} 0 & \delta_{ij} w_{ir}^{(i)} \\ \delta_{ij} w_{is}^{(i)} & w_{ir,\,js}^{(i)} \end{bmatrix}, \quad (i, j = 1, 2 \ldots K;\ r, s = 1, 2 \ldots N).$$

The inverse of the matrix W (28) is again a compound matrix:

$$(30)\quad W^{-1} = [B^{(ij)}].$$

Now we obtain:

$$(31)\quad \partial x_{ks}/\partial M_t = B_{ks}^{(tk)} \cdot L_t.$$

The income elasticity is:

$$(32)\quad Ex_{ks}/EM_t = B_{ks}^{(tk)} \cdot L_t \cdot M_t/x_{ks}.$$

We obtain also:

$$(33)\quad \partial x_{ks}/\partial p_r = -\sum_{i=1}^{K} x_{ir} B_{is}^{(ik)} L_i + \sum_{i=1}^{K} L_i B_{ir,\,is}^{(ik)}$$

and for the price elasticity:

$$(34)\quad Ex_{ks}/Ep_r = \sum_{i=1}^{K} -m_{ir}(Ex_{is}/EM_i) + m_{ir}\sigma_{ir,\,is}^{(k)} \cdot (x_{is}/x_{ks}).$$

In formula (34) we denote by $\sigma_{ir,\,is}^{(k)}$ the partial elasticity of individual k for the substitution of commodity r for commodity s by the individual i.

For the market demand, we obtain the income elasticity:

$$(35)\quad EX_s/EM_t = \sum_{k=1}^{K} (Ex_{ks}/EM_t)$$

and for the price elasticity:

$$(36)\quad EX_s/Ep_r = \sum_{k=1}^{K} \sum_{i=1}^{K} -m_{ir}(Ex_{is}/EM_i) + m_{ir}\sigma_{ir,\,is}^{(k)} (x_{is}/x_{ks}).$$

The methods used by Wald to derive empirical indifference

systems[8]) and the recent studies based upon household budget surveys[9]) give us hope that an empirical, econometric verification of the theory may be possible.

[8]) A. Wald, "The Approximate Determination of Indifference Systems by Means of Engel Curves," *Econometrica*, vol. 8 (1940) 144 ff. See also the results of J. A. Nordin in G. Tintner, *Econometrics* (New York: Wiley, 1952), pp. 60 ff.

[9]) R. Stone, D. A. Rowe, W. J. Corlett, R. Hurstfield, and M. Potter, *The Measurement of Consumer's Expenditure and Behaviour in the United Kingdom*, 1920—38 (Cambridge: Cambridge University Press, 1954) H. Wold and L. Juréen, *Demand Analysis* (New York: Wiley, 1953). J. S. Prais and H. S. Houthakker, *op. cit.*

RALPH W. PFOUTS

Hours of Work, Savings and the Utility Function

The contributions of Harold Hotelling to the theory of utility and demand place him in a small group of pioneers that altered the fundamentals of economic theory. His work, coming at about the same time as that of Hicks, Allen and Schultz, and coinciding approximately with the discovery of Slutsky's work, helped to lay to rest older concepts of measurable utility, to unite utility and demand theory and to rid the theory of unnecessary *ceteris paribus* assumptions. Through these advances a coherent theory of utility and demand was provided for the segments of economics that must rely on utility and demand theory.

The purpose of this paper is to present a modest extension of this utility and demand theory that has become the present-day orthodoxy. Specifically, the suggestion will be made that hours of work and savings are variables affecting psychological values and are deserving of consideration to the same extent as the quantities of the various goods consumed.

I

The Assumptions

If hours of work and savings are inserted into the utility function, several additional results are achieved that cannot be deduced in the case in which only the quantities of the goods appear as independent variables. In the case that includes hours of work and savings in the utility function, a more elaborate and more realistic prototype of the consumer is available, and monetary considerations can be brought into the pattern of consumer behavior. This suggests that the more elaborate case may yield a consumer element suitable for

aggregation into marginal propensity to save or marginal propensity to consumer schedules, thus bringing micro-theories and macro-theories into closer alignment. Furthermore, new demand restrictions, in addition to modification of known restrictions, are obtained from this formulation.

In this paper we accept without extensive justification the desirability of including hours of work and savings in the utility function. In any case, this seems unnecessary because discussions of demand for leisure can be found in economic literature dating from the time of the classical economists.[1]) It would seem clear and acceptable that hours of work affect both the individual's level of satisfaction and his income.

Similarly there seems to be little need for justifying the inclusion of savings in the utility function. If savings were not actively desired by individuals, persons in the lower and middle income groups would spend their money on goods and services entirely, and savings would be accumulated only by those with such large incomes that they could not spend the entire amounts on consumers' goods and services. As Lord Keynes put it, an act of saving by an individual "means — so to speak — a decision not to have dinner today."[2])

Thus we write the utility function as

$$U = \Phi(x_1, \ldots, x_n, y, s), \tag{1}$$

where the x's indicate quantities of goods and services, y represents hours of work and s stands for current savings in dollars. We assume that

[1]) Among the more noted and more recent writings on this subject must be listed: A. C. Pigou, *The Economics of Welfare* (London, 1929), p. 573, and *A Study in Public Finance*, (London, 1928), pp. 83—84. F. H. Knight, *Risk, Uncertainty and Profit* (New York and Boston, 1921), p. 117. L. Robbins, "On the Elasticity of Demand for Income in Terms of Effort," *Economica*, X (1930), 123—29, reprinted in A. E. A. *Readings in the Theory of Income Distribution*, W. Fellner and B. F. Haley, eds. (Philadelphia and Toronto, 1946), pp. 237—44. F. W. Paish, "Economic Incentive in Wartime," *Economica*, N. S. VIII (1941), 239—48. G. Cooper, "Taxation and Incentive in Mobilization," *Quarterly Journal of Economics*, LXVI (1952), 43—66.

[2]) Keynes, *The General Theory of Employment, Interest and Money* (London and New York, 1936), pp. 19, 21, 64—65, 210—13.

$$\frac{\partial U}{\partial x_i} = \Phi_i > 0,$$

$$\frac{\partial U}{\partial y} = \Phi_y < 0,$$

$$\frac{\partial U}{\partial s} = \Phi_s > 0.$$

In other words goods and savings have utility; work, at least after a certain time, has disutility.

Since current savings are introduced into the utility function, a stock of earning assets must be considered in the budget equation as a source of income. In addition, since hours of work are included in the utility function, it is appropriate to include the wage rate in the budget equation. As a consequence the budget equation is

$$\sum_i p_i x_i + s = wy + rM. \tag{2}$$

In this equation the p's are the prices of the goods and services, w is the wage rate, M is the stock of earning assets and r is the rate of return. The p's, w and r are assumed to be mutually independent variables. Clearly M includes the accumulation of savings from past periods,, but it may also include windfalls.[3])

There are advantages to be obtained from writing the utility equation in the form of (1) and the budget equation in the form of (2) that we will note only briefly. In the first place these equations specifically include hours of work and consequently should be readily usable in welfare economics. That is, an individual's economic welfare depends not only the income he receives, or the goods and services he purchases, but also on the hours of work he must render to obtain the income, or the goods and services. Through introducing hours of work directly into the equations that determine the consumer's equilibrium

[3]) By using rM in (2) we implicitly assume either that the individual puts all his earning assets in uses showing the same rate of return or that r is an average rate of return. It would be more realistic to write $\sum r_i M_i$ in place of rM to show a differentiated portfolio. This would get into complications that it seems desirable to avoid for purposes of the present paper i.e., we avoid all problems of expectations, principles of portfolio selection, etc.

we introduce directly an important component of welfare that previously was ignored or introduced by indirection.

The usual budget equation can be modified to include hours of work by treating work as a negative good. The budget restriction then is

$$\sum_i p_i x_i = 0.$$

The disadvantage of this equation, beside appearing unnatural, is that there is no fixed income exhibited as in the common form of the budget equation and there is no term corresponding to rM in (2). As a result the income effect cannot be readily identified if resort is made the treating work as a negative good. Hence the advantage of introducing hours of work is offset by obscuring the income effect. This difficulty does not arise in the formulation advanced below.

The Equilibrium

Then to find the consumer's equilibrium position under the assumptions outlined above, we form the equation

$$(3) \quad \Lambda = \Phi(x_i, \ldots, x_n, y, s) + \lambda\left(\sum_i p_i x_i + s - wy - rM\right).$$

This yields equations that, together with (2), describe the maximum satisfaction position for the consumer:

$$(4) \quad \begin{aligned} \frac{\partial \Lambda}{\partial x_i} &= \Phi_i + \lambda p_i = 0, \qquad (i = 1, \ldots, n), \\ \frac{\partial \Lambda}{\partial y} &= \Phi_y - \lambda w = 0, \\ \frac{\partial \Lambda}{\partial s} &= \Phi_s + \lambda = 0. \end{aligned}$$

Equations (4) and equation (2) are clearly analogous to the orthodox method of determining the consumer's equilibrium position.

II

The Effects of Price Changes

As would be expected from recalling utility theory that does

not include hours of work and savings, the most interesting points arise from consideration of displacements from the equilibrium condition. Thus a change in the price of the kth good leads to the equations

$$\sum_j \Phi_{ij} \frac{\partial x_j}{\partial p_k} + \Phi_{iy} \frac{\partial y}{\partial p_k} + \Phi_{is} \frac{\partial s}{\partial p_k} + p_i \frac{\partial \lambda}{\partial p_k} = -\lambda \delta_{ik} \tag{5}$$

where δ_{ik} is the Kronecker delta and $i = 1, \ldots, n$.

In addition we have

$$\begin{aligned} &\sum_j \Phi_{yj} \frac{\partial x_j}{\partial p_k} + \Phi_{yy} \frac{\partial y}{\partial p_k} + \Phi_{ys} \frac{\partial s}{\partial p_k} - w \frac{\partial \lambda}{\partial p_k} = 0, \\ &\sum_j \Phi_{sj} \frac{\partial x_j}{\partial p_k} + \Phi_{sy} \frac{\partial y}{\partial p_k} + \Phi_{ss} \frac{\partial s}{\partial p_k} + \frac{\partial \lambda}{\partial p_k} = 0, \\ &\sum_j p_j \frac{\partial x_j}{\partial p_k} - w \frac{\partial y}{\partial p_k} + \frac{\partial s}{\partial p_k} = -x_k. \end{aligned} \tag{6}$$

Then proceeding in the orthodox manner we make use of the determinant

$$D = \begin{vmatrix} \Phi_{11} \ldots \Phi_{1n} & \Phi_{1y} & \Phi_{1s} & p_1 \\ \vdots & & & \\ \Phi_{n1} \ldots \Phi_{nn} & \Phi_{ny} & \Phi_{ns} & p_n \\ \Phi_{y1} \ldots \Phi_{yn} & \Phi_{yy} & \Phi_{ys} & -w \\ \Phi_{s1} \ldots \Phi_{sn} & \Phi_{sy} & \Phi_{ss} & 1 \\ p_1 \ldots p_n & -w & 1 & 0 \end{vmatrix}, \tag{7}$$

and Cramer's rule to obtain

$$\frac{\partial x_i}{\partial p_k} = -\lambda \frac{D_{ki}}{D} - x_k \frac{D_{n+3,i}}{D}, \qquad (i = 1, \ldots, n), \tag{8}$$

$$\frac{\partial y}{\partial p_k} = -\lambda \frac{D_{k,n+1}}{D} - x_k \frac{D_{n+3,n+1}}{D}, \tag{9}$$

$$\frac{\partial s}{\partial p_k} = -\lambda \frac{D_{k,n+2}}{D} - x_k \frac{D_{n+3,n+2}}{D}. \tag{10}$$

In (8), (9) and (10) the subscripts to D indicate cofactors. The

last three equations are clearly comparable to the Slutsky-Hicks equation.[4] They show the effect of a price change on the quantities of the goods purchased by the individual, the hours of work the individual wishes to offer and the savings that he wants to lay by. In each case the effect of the price change is divided into two parts, again obviously comparable to the substitution and income effects of the Slutsky-Hicks equation.

To continue the parallel with the Slutsky-Hicks equation, we examine the income effects directly. Under the assumptions that we are using, there is no single income effect, but rather income effects arising from either a change in the rate of return or a change in the stock of earning assets. Thus if we differentiate equations (4) and equation (2) with respect to r, the rate of return, we get

$$(11)\quad \begin{aligned} &\sum_j \Phi_{ij}\frac{\partial x_j}{\partial r} + \Phi_{iy}\frac{\partial y}{\partial r} + \Phi_{is}\frac{\partial s}{\partial r} + p_i\frac{\partial \lambda}{\partial r} = 0, \quad (i = 1, \ldots, n) \\ &\sum_j \Phi_{yj}\frac{\partial x_j}{\partial r} + \Phi_{yy}\frac{\partial y}{\partial r} + \Phi_{ys}\frac{\partial s}{\partial r} - w\frac{\partial \lambda}{\partial r} = 0, \\ &\sum_j \Phi_{sj}\frac{\partial x_j}{\partial r} + \Phi_{sy}\frac{\partial y}{\partial r} + \Phi_{ss}\frac{\partial s}{\partial r} + \frac{\partial \lambda}{\partial r} = 0, \\ &\sum_j p_j\frac{\partial x_j}{\partial r} - w\frac{\partial y}{\partial r} + \frac{\partial s}{\partial r} = M. \end{aligned}$$

Solving (11) yields

$$(12)\qquad \frac{\partial x_i}{\partial r} = M\frac{D_{n+3,i}}{D}, \qquad (i = 1, \ldots, n),$$

$$(13)\qquad \frac{\partial y}{\partial r} = M\frac{D_{n+3,n+1}}{D},$$

$$(14)\qquad \frac{\partial s}{\partial r} = M\frac{D_{n+3,n+2}}{D}.$$

Then substituting from (12) to (8), from (13) to (9) and from (14) to (10) gives us

$$(15)\qquad \frac{\partial x_i}{\partial p_k} = -\lambda\frac{D_{ki}}{D} - \frac{x_k}{M}\frac{\partial x_i}{\partial r}, \qquad (i = 1, \ldots, n)$$

[4]) See, for example, J. R. Hicks, *Value and Capital* (2nd ed., Oxford, 1946) pp. 306—9.

$$\frac{\partial y}{\partial p_k} = -\lambda \frac{D_{k,n+1}}{D} - \frac{x_k}{M} \frac{\partial y}{\partial r}, \tag{16}$$

$$\frac{\partial s}{\partial p_k} = -\lambda \frac{D_{k,n+2}}{D} - \frac{x_k}{M} \frac{\partial s}{\partial r}. \tag{17}$$

In equations (15), (16) and (17) the effects of a price change on the quantity of a good, the hours of work and savings are divided into a substitution effect and an income effect, the income effect resulting from a change in the rate of return. We can obtain equations similar to (15), (16) and (17) by considering the effects of a change in M, the stock of earning assets. To do this we differentiate equations (4) and equation (2) with respect to M obtaining

$$\begin{aligned}
&\sum_j \Phi_{ij} \frac{\partial x_j}{\partial M} + \Phi_{iy} \frac{\partial y}{\partial M} + \Phi_{is} \frac{\partial s}{\partial M} + p_i \frac{\partial \lambda}{\partial M} = 0, \quad (i = 1, \ldots, n)\\
&\sum_j \Phi_{yj} \frac{\partial x_j}{\partial M} + \Phi_{yy} \frac{\partial y}{\partial M} + \Phi_{ys} \frac{\partial s}{\partial M} - w \frac{\partial \lambda}{\partial M} = 0,\\
&\sum_j \Phi_{sj} \frac{\partial x_j}{\partial M} + \Phi_{sy} \frac{\partial y}{\partial M} + \Phi_{ss} \frac{\partial s}{\partial M} + \frac{\partial \lambda}{\partial M} = 0,\\
&\sum_j p_j \frac{\partial x_j}{\partial M} - w \frac{\partial y}{\partial M} + \frac{\partial s}{\partial M} = r.
\end{aligned} \tag{18}$$

Solving equations (18) yields

$$\frac{\partial x_i}{\partial M} = r \frac{D_{n+3,i}}{D}, \qquad (i = 1, \ldots, n), \tag{19}$$

$$\frac{\partial y}{\partial M} = r \frac{D_{n+3,n+1}}{D}, \tag{20}$$

$$\frac{\partial s}{\partial M} = r \frac{D_{n+3,n+2}}{D}. \tag{21}$$

Then substituting from (19) to (8), from (20) to (9) and from (21) to (10) shows

$$\frac{\partial x_i}{\partial p_k} = -\lambda \frac{D_{ki}}{D} - \frac{x_k}{r} \frac{\partial x_i}{\partial M}, \tag{22}$$

$$\frac{\partial y}{\partial p_k} = -\lambda \frac{D_{k,n+1}}{D} - \frac{x_k}{r}\frac{\partial y}{\partial M}, \tag{23}$$

$$\frac{\partial s}{\partial p_k} = -\lambda \frac{D_{k,n+2}}{D} - \frac{x_k}{r}\frac{\partial s}{\partial M}. \tag{24}$$

Inspection of equations (22), (23) and (24) again shows the effect of a price change on the quantity of a good, hours of work, and saving divided into a substitution effect and an income effect, the income effect now being related to a change in the stock of earning assets. Again the similarity to the Slutsky-Hicks equation is obvious.

The Effects of a Wage Rate Change

A displacement from equilibrium due to a change in the wage rate also shows a compound effect similar to that of the Slutsky-Hicks equation. To approach this we find the partial derivatives of equations (4) and equation (2) with respect to the wage rate, obtaining

$$\begin{aligned}
&\sum_j \Phi_{ij}\frac{\partial x_j}{\partial w} + \Phi_{iy}\frac{\partial y}{\partial w} + \Phi_{is}\frac{\partial s}{\partial w} + p_i\frac{\partial \lambda}{\partial w} = 0, \quad (i = 1, \ldots, n),\\
&\sum_j \Phi_{yj}\frac{\partial x_j}{\partial w} + \Phi_{yy}\frac{\partial y}{\partial w} + \Phi_{ys}\frac{\partial s}{\partial w} - w\frac{\partial \lambda}{\partial w} = \lambda,\\
&\sum_j \Phi_{sj}\frac{\partial x_j}{\partial w} + \Phi_{sy}\frac{\partial y}{\partial w} + \Phi_{ss}\frac{\partial s}{\partial w} + \frac{\partial \lambda}{\partial w} = 0,\\
&\sum_j p_j\frac{\partial x_j}{\partial w} - w\frac{\partial y}{\partial w} + \frac{\partial s}{\partial w} = y.
\end{aligned} \tag{25}$$

Solving equations (25) shows us that

$$\frac{\partial x_i}{\partial w} = \lambda\frac{D_{n+1,i}}{D} + y\frac{D_{n+3,i}}{D}, \qquad (i = 1, \ldots, n) \tag{26}$$

$$\frac{\partial y}{\partial w} = \lambda\frac{D_{n+1,n+1}}{D} + y\frac{D_{n+3,n+1}}{D}, \tag{27}$$

$$\frac{\partial s}{\partial w} = \lambda\frac{D_{n+1,n+2}}{D} + y\frac{D_{n+3,n+2}}{D}. \tag{28}$$

The similarity of equations (26), (27) and (28) to the Slutsky-Hicks equation is obvious and, as in the case of a response to a price change, we have a choice of two income effects. Thus we can substitute from (12) to (26), from (13) to (27) and from (14) to (28) obtaining

$$\frac{\partial x_i}{\partial w} = \lambda \frac{D_{n+1,i}}{D} + \frac{y}{M}\frac{\partial x_i}{\partial r}, \qquad (i = 1, \ldots, n), \tag{29}$$

$$\frac{\partial y}{\partial w} = \lambda \frac{D_{n+1,n+1}}{D} + \frac{y}{M}\frac{\partial y}{\partial r}, \tag{30}$$

$$\frac{\partial s}{\partial w} = \lambda \frac{D_{n+1,n+2}}{D} + \frac{y}{M}\frac{\partial s}{\partial r}. \tag{31}$$

Alternatively we can substitute from (19) to (26), from (20) to (27) and from (21) to (28); this yields

$$\frac{\partial x_i}{\partial w} = \lambda \frac{D_{n+1,i}}{D} + \frac{y}{r}\frac{\partial x_i}{\partial M}, \qquad (i = 1, \ldots, n), \tag{32}$$

$$\frac{\partial y}{\partial w} = \lambda \frac{D_{n+1,n+1}}{D} + \frac{y}{r}\frac{\partial y}{\partial M}, \tag{33}$$

$$\frac{\partial s}{\partial w} = \lambda \frac{D_{n+1,n+2}}{D} + \frac{y}{r}\frac{\partial s}{\partial M}. \tag{34}$$

In these latter equations the income effect shows the result of a change in the stock of earning assets.

Restrictions on Consumer Behavior

For many years critics of utility and demand theory have argued that this theory was too restrictive to constitute a reasonable description of human behavior. Basically the argument contends that, in the theory, the path of a consumer's behavior is preordained and that he responds too mechanically to changes in prices and income; that too few allowances for individual deviations are made; in short that he is Veblen's "oscillating globule of desire." In considering the validity of this criticism it is instructive to consider the changes in quantities of goods, hours of work and savings as developed in this paper.

It is quite literally true that in no case developed in this paper can we be certain of the direction of change of quantities of goods, hours of work or savings in response to a change in prices, wage rate, rate of return or stock of earning assets. That is, on the basis of logic or theory alone, we cannot be certain that a change in any dependent variable with respect to any independent variable is invariably positive or negative. For such conclusions we must resort to empirical observation. Surely this is not an overly restrictive representation of consumer behavior.

Indeed in only two of the cases considered does the mathematics show any indication of the direction of change. One of these is the well known instance:

$$\frac{\partial x_i}{\partial p_i} = -\lambda \frac{D_{ii}}{D} - \frac{x_i}{r} \frac{\partial x_i}{\partial M},$$

i.e. the case of the response of the quantity of a good to a change in its own price. In this equation the first term of the righthand member must be negative. This is true because the principle minors of the determinant in (7) oscillate in sign, hence the ratio of cofactor to determinant must be negative, while $-\lambda$ must be positive. But this is not sufficient to insure a negative response of the good to a change in its own price because the second term of the righthand member could be positive and larger in absolute amount than the first term.

The second instance in which the mathematics gives some indication of the direction of change is the case of the response of hours of work to change in the wage rate. The equation in this case is of the type

$$\frac{\partial y}{\partial w} = \lambda \frac{D_{n+1,\, n+1}}{D} + \frac{y}{r} \frac{\partial y}{\partial M}.$$

In this equation the first term of the righthand member is obviously positive, but again this is not sufficient to assure that hours of work increase in response to a change in wage rate because the income term can be sufficiently negative to offset the other term. Indeed, since the income term represents a change in unearned income, it seems quite probable on empirical

grounds that the income term will be negative and quite large. Consequently we cannot be certain of the direction of change in this case either, or, indeed, in any case without reference to empirical observation.

III

Some Characteristics of the Demand Equation

The equilibrium conditions, equations (4) and (2), represent $n + 3$ conditions on the $2n + 6$ variables, $x_1, \ldots, x_n; p_1, \ldots, p_n;$ y, s, w, r, M and λ. Then assuming that there are no mathematical difficulties, it is possible to solve for $n + 3$ of the variables in terms of the remainder.[5]) It is of special interest to view these solutions as demand equations written as

$$x_i = h_i\,(p_1, \ldots, p_n, M, r, w), \quad (i = 1, \ldots, n) \tag{35}$$

$$y = h_y(p_1, \ldots, p_n, M, r, w), \tag{36}$$

$$s = h_s(p_1, \ldots, p_n, M, r, w), \tag{37}$$

$$-\lambda = h_\lambda(p_1, \ldots, p_n, M, r, w). \tag{38}$$

In this form the equations show as dependent variables the quantities that are actually subject to control by the individual.

The demand equations that we have formulated are free of any particular index, i.e., it does not make any difference what utility index we use. Thus the situation with respect to the choice of utility indicator is the same as in the case in which savings and hours of work are not included in the utility function. The freedom from the utility index may be seen by rewriting equations (4) as

$$-\lambda = \frac{\Phi_i}{p_i} = \frac{-\Phi_y}{w} = \Phi_s, \qquad (i = 1, \ldots, n). \tag{39}$$

This shows, among other things, that the marginal utility of money is equal to the marginal utility of savings in the equilibrium position. From equations (39) we obtain

[5]) See P. A. Samuelson, *Foundations of Economic Analysis* (Cambridge, Mass., 1947), pp. 99—100.

$$
\begin{aligned}
&\frac{\Phi_i}{\Phi_j} = \frac{p_i}{p_j} = \frac{F'\Phi_i}{F'\Phi_j},\\
&\frac{\Phi_i}{-\Phi_y} = \frac{p_i}{w} = \frac{F'\Phi_i}{-F'\Phi_y},\\
(40)\qquad &\frac{\Phi_i}{\Phi_s} = p_i = \frac{F'\Phi_i}{F'\Phi_s},\\
&\frac{-\Phi_y}{\Phi_s} = w = \frac{-F'\Phi_y}{F'\Phi_s},
\end{aligned}
$$

where F is a monotonic transformation. Equations (40) show that if a monotonic transformation of the utility function were made, it would have no effect on the equilibrium position.[6])

The Absence of a Homogeneity Condition

In the orthodox development of the theory of individual demand, the demand equations are homogeneous of order zero with respect to prices and income.[7]) This arises from the absence of savings in the budget equation and from equilibrium conditions similar to the first equations of (40). In the formulation developed in this paper, the individual demand equations are no longer homogeneous of order zero with respect to the independent variables. This may be seen from inspection of equation (2). Since savings have no price in the ordinary sense, a proportional change in the prices associated with other variables does not affect the term representing savings. Consequently the equality in (2) is not maintained. Similarly, from inspection of the last two equations in (40), it is clear that a proportional change in prices and wage rate will alter the equilibrium position. Consequently, the restriction that the demand equations be homogeneous of order zero is lost in the present formulation. It is interesting to notice that this restriction is preserved if hours of work are put into the utility equation and savings are not.[8])

[6]) *Ibid.*, pp. 97—99.

[7]) *Ibid.*, pp. 104—7.

[8]) See F. Gilbert and R. W. Pfouts, "A Theory of the Responsiveness of Hours of Work to Changes in the Wage Rate," *Review of Economics and Statistics*, XL (1958), pp. 116—21.

The point of the preceding paragraph can be developed more finely by considering the equation

$$
\begin{aligned}
&\sum_k p_k \frac{\partial x_i}{\partial p_k} + M\frac{\partial x_i}{\partial M} + r\frac{\partial x_i}{\partial r} + w\frac{\partial x_i}{\partial w} \\
(41) \qquad &= -\sum_k p_k \lambda \frac{D_{ki}}{D} - \sum_k p_k x_k \frac{D_{n+3,i}}{D} + rM\frac{D_{n+3,i}}{D} \\
&+ rM\frac{D_{n+3,i}}{D} + w\left(\lambda\frac{D_{n+1,i}}{D} + y\frac{D_{n+3,i}}{D}\right),
\end{aligned}
$$

the substitutions being made from equations (8), (19), (12) and (26). If the homogeneity condition held in the present formulation, the left member of (41) would be identically zero by Euler's theorem on homogeneous functions; we shall see that this is not true in general. If we add and substract $\lambda(D_{n+2,i}/D)$ to the right member of (41) and rearrange terms we obtain

$$
\begin{aligned}
&-\sum_k p_k \lambda \frac{D_{ki}}{D} + w\lambda\frac{D_{n+1,i}}{D} - \lambda\frac{D_{n+2,i}}{D} \\
(42) \qquad &-\sum_k p_k x_k \frac{D_{n+3,i}}{D} + rM\frac{D_{n+3,i}}{D} + wy\frac{D_{n+3,i}}{D} \\
&+ rM\frac{D_{n+3,i}}{D} + \lambda\frac{D_{n+2,i}}{D}.
\end{aligned}
$$

The first three terms in expression (42) can be eliminated by reference to (7) and the theorem that expansion of a determinant by the elements of one column and the corresponding cofactors of another column yields zero. By reference to (2) it follows that the second three terms of (42) reduce to $s(D_{n+3,i}/D)$. Consequently we can write

$$
\begin{aligned}
&\sum_k p_k \frac{\partial x_i}{\partial p_k} + M\frac{\partial x_i}{\partial M} + r\frac{\partial x_i}{\partial r} + w\frac{\partial x_i}{\partial w} \\
(43) \qquad &= s\frac{D_{n+3,i}}{D} + rM\frac{D_{n+3,i}}{D} + \lambda\frac{D_{n+2,i}}{D}.
\end{aligned}
$$

Substituting from (19) and (24) gives

$$(44) \qquad \sum_k p_k \frac{\partial x_i}{\partial p_k} + r\frac{\partial x_i}{\partial r} + w\frac{\partial x_i}{\partial w} = \frac{s}{r}\frac{\partial x_i}{\partial M} - \frac{x_i}{r}\frac{\partial s}{\partial M} - \frac{\partial s}{\partial p_i}.$$

If (44) is transformed into elasticity form, we obtain

$$(45) \qquad \sum_k \eta_{ik} + \eta_{ir} + \eta_{iw} = s\left(\frac{1}{rM}(\eta_{iM} - \eta_{sM}) - \frac{1}{p_i x_i}\eta_{si}\right).$$

While (44) and (45) lack the straight-forward simplicity of the homogeneity condition, they do bring out certain points. First it may be noted by reference to these equations that the homogeneity condition clearly does not hold under the present assumptions; further it is difficult to visualize an assumption with intuitive appeal that would restore the homogeneity condition. Secondly, the influence of savings in upsetting the homogeneity condition is clearly exhibited in the righthand members of these equations. Observation of the righthand member of (45) shows that if savings were zero, the homogeneity condition would almost be restored. But the full homogeneity condition would not be restored because the elasticity of the i^{th} good with respect to the stock of earning assets would still be missing from the lefthand member of (45).

A similar development can be made for equation (36), the equation expressing the demand of the individual for hours of work. In elasticity from the result is

$$(46) \qquad \sum_k \eta_{yk} + \eta_{yr} + \eta_{yw} = s\left[\frac{1}{rM}(\eta_{yM} - \eta_{sM}) + \frac{1}{yw}\eta_{sw}\right].$$

Reference to (46) shows that the homogeneity condition does not hold in general for (36). Again the importance of savings in removing the homogeneity condition with reference to the demand for hours of work is brought out in the righthand member of (46).

In the same fashion equation (37) can be used to show the absence of the homogeneity condition in the demand for savings. When exhibited in elasticity form the result is

$$(47) \qquad \sum_k \eta_{sk} + \eta_{sr} + \eta_{sw} - \frac{s}{rM}\eta_{sM} = \frac{\lambda}{s}\frac{D_{n+2,\, n+2}}{D}.$$

Again the homogeneity condition is clearly not satisfied. With

regard to this righthand member of (47) it may be observed that the ratio of cofactor to determinant will be negative since principal minors of the determinant in (7) will alternate in sign; at the same time λ will be negative. Consequently the sign of each member is the same as the sign of s. If savings are positive each member of (47) is positive, but if dissaving takes place each member is negative.

Effects of Saving and Dissaving

By making certain assumptions that appear to be reasonable on empirical grounds, we can obtain some additional results from (47). We assume that the elasticity of savings with respect to price is negative but that the elasticities of savings with respect to wage rate, rate of return and stock of earning assets are positive. It follows from (47) that if

$$s > 0,$$

then

$$\eta_{sr} + \eta_{sw} - \frac{s}{rM}\eta_{sM} > -\sum_k \eta_{sk}. \tag{48}$$

Under the assumptions about the signs of the elasticities, the first two terms in the lefthand member of (48) are both positive and the third term has a negative effect, while the right member is positive, consisting of a negative term prefixed by a minus sign. Then (48) implies that savings respond more heavily to changes in rate of return and in wage rate than they do to prices and to stock of earning assets weighted by savings as a proportion of income from earning assets. In other words, rate of return and wage rate dominate the weighted value of stock of earning assets and prices in affecting savings.

It also follows from (47), that if

$$s < 0,$$

then

$$\eta_{sw} + \eta_{sr} - \frac{s}{rM}\eta_{sM} < -\sum_k \eta_{sk}. \tag{49}$$

Continuing the assumptions concerning the signs of the elasticities of savings, the first two terms of (49) are positive, the third is

positive but prefixed by a minus sign and the last term is negative and prefixed by a minus sign. Under these conditions it follows that dissavings respond more strongly to price changes than to changes in wage rate, rate of return and stock of earning assets with the appropriate weight. Consequently when dissaving takes place, price influences dominate all other influences. Surely this has a strong intuitive appeal since one would expect dissaving to take place at an income level so low that prices of consumers' goods dominated all consumers' actions.

IV

New and Modified Slutsky Conditions

The Slutsky conditions, which constitute restrictions on the demand equations, are modified from their usual form when hours of work and savings are included in the utility function. This is easily seen by reference to equation (15), keeping in mind the symmetry of D. From (15) we may write

$$M\frac{\partial x_i}{\partial p_k} + x_k\frac{\partial x_i}{\partial r} = M\frac{\partial x_k}{\partial p_i} + x_i\frac{\partial x_k}{\partial r}. \tag{50}$$

And from (22) we may write

$$r\frac{\partial x_i}{\partial p_k} + x_k\frac{\partial x_i}{\partial M} = r\frac{\partial x_k}{\partial p_i} + x_i\frac{\partial x_k}{\partial M}. \tag{51}$$

In addition to the modified Slutsky conditions of (50) and (51), we can find another restriction on the demand equations similar to the Slutsky conditions. From comparing equations (16) and equation (29), we see that it is possible to write

$$M\frac{\partial x_i}{\partial w} - y\frac{\partial x_i}{\partial r} = -M\frac{\partial y}{\partial p_i} - x_i\frac{\partial y}{\partial r}. \tag{52}$$

Equation (52) is a Slutsky-type restriction and presumably under ideal conditions would be capable of being tested for refutation. Alternatively, from equations (23) and (32) it is easily seen that we can say

$$r\frac{\partial x_i}{\partial w} - y\frac{\partial x_i}{\partial M} = -r\frac{\partial y}{\partial p_i} - x_i\frac{\partial y}{\partial M}. \tag{53}$$

Some Restrictions on Elasticities

In addition to the Slutsky-type of restriction, we can bring forward other restrictions concerning the elasticity of the dependent variables with respect to rate of return and the stock of earning assets. Put briefly, this class of restrictions says that the elasticity of any dependent variable, i.e. the x's, y or s, with respect to the rate of return equals the elasticity of the same variable with respect to the stock of earning assets.

Thus from equations (12) and (19) we notice that

$$\frac{1}{r}\frac{\partial x_i}{\partial M} = \frac{1}{M}\frac{\partial x_i}{\partial r};$$

consequently

$$\frac{M}{x_i}\frac{\partial x_i}{\partial M} = \frac{r}{x_i}\frac{\partial x_i}{\partial r} = \eta_{iM} = \eta_{ir}, \quad (i = 1, \ldots, n). \tag{54}$$

From a comparison of (13) and (20) it is clear that we can also state that

$$\eta_{yr} = \eta_{yM}. \tag{55}$$

It is equally obvious from (14) and (21) that

$$\eta_{sr} = \eta_{sM}. \tag{56}$$

The restrictions on the demand equations that are represented by equations (54), (55) and (56) are restrictions on individual demand equations at the same level as the Slutsky conditions. I.e. they arise from the fundamental assumptions regarding the utility equation and the postulate of maximization. In an empirical sense they are both less complex and, probably, less important than the Slutsky conditions; empirically they appear to be less restrictive on the variables than the Slutsky conditions. In substance they merely state that the effect of a change in income from earning assets on the quantities of goods, hours of work or savings is the same whether the increment results from a change in rate of return or a change in stock of earning assets.

The Micro-Propensity to Save

By employing hours of work and savings as variables in the utility function, we gain a more realistic prototype of the

consumer and additional restrictions on the individual demand equations. In addition the marginal propensity to save takes on meaning in the case of an individual. That is, we can now write

$$ds = \frac{\partial s}{\partial w} dw + \frac{\partial s}{\partial r} dr + \frac{\partial s}{\partial M} dM + \sum_i \frac{\partial s}{\partial p_i} dp_i \tag{57}$$

as a meaningful statement in which substitutions can be made and which is related to other aspects of the consumer's behavior. An individual propensity to save relation is not possible in the orthodox formulation of utility.

Indeed the micro-approach yields a more comprehensive statement of the motives behind the marginal propensity to save because it includes the effects of relative prices in the last term of the right member of (57). That is, the macro-approach usually shows the propensity to save as a function of income, but not of relative prices. The response of savings to changes in income, but not prices, at the micro level is given by

$$ds = \frac{\partial s}{\partial w} dw + \frac{\partial s}{\partial r} dr + \frac{\partial s}{\partial M} dM. \tag{58}$$

But equation (58) is strictly valid only if there have been no changes in the prices of the goods consumed, or if the last term of (57), through a coincidence, is equal to zero. The absence of *general* validity for equation (58), coupled with the fact that the micro propensity to save is a combination of many micro propensities to save, suggests that a part of the difficulties in empirical studies of the macro propensity to save or to consume may be due to their ignoring relative prices as independent variables.

PART III

On Economic Dynamics

MILTON FRIEDMAN

In Defence of Destabilizing Speculation[1]

Two propositions about private speculation are widely held: first, that speculation is in fact often destabilizing, in the sense that it makes fluctuations in prices wider than they would "otherwise" be; second, that destabilizing speculation necessarily involves economic loss. This pair of propositions underlies much current opinion about commodity policy — where they lead to support for "buffer stocks" and similar plans — and about balance of payments policy — where they constitute a chief criticism of floating exchange rates.

This note is not intended to be an exhaustive analysis of this pair of propositions, or of speculation in general. Its purpose is much more limited: to point out that the second proposition is invalid, that destabilizing speculation, though it may in some cases lead to economic loss, may in others confer economic benefit. The empirical generalization about the prevalence of destabilizing speculation — which is what gives the theoretical proposition its interest — seems to be one of those propositions that has gained currency the way a rumor does — each man believes it because the next man does — and despite the absence of any substantial body of well documented evidence for it. It is a proposition that badly needs intensive empirical investigation. My own conjecture is that such an investigation would show it to be unfounded. But this is simply a conjecture and plays no part in what follows.

The ready acceptance of the proposition that destabilizing speculation is economically harmful reflects, I believe, a natural bias of the academic student against gambling and in favor of insurance. It is natural for him to regard a futures market, for example, as a market in which a "legitimate" producer hedges

[1]) I am indebted for comments on an earlier draft to Martin Bailey, Harry Johnson, James Meade, Joan Robinson, and Dennis Robertson.

his risks by transferring them to a "speculator"; the producer is viewed as buying "insurance" from the speculator. But granted that this is a possible and indeed likely interpretation of an actual futures market, it is not the only possible one. May such a market not be one in which the "legitimate" producer engages as a side-line in selling "gambles" to speculators willing to pay a price for gambling and knowingly doing so? And if so, moral scruples about gambling aside, is any economic loss involved?

In arguing that destabilizing speculation need not involve economic loss I do not mean in any way to deny the usual view that stabilizing speculation confers benefit. In this usual view, the economic function of speculation is taken to be the reduction of inter-temporal differences in price. In a commodity market, for example, a speculator is viewed as performing this function by buying when the crop is plentiful and prices "abnormally" low, holding stocks of the commodity until prices have risen, and then selling when the crop is short and prices "abnormally" high. In this way, speculators transfer resources from less to more urgent uses. The difference between the prices at which they sell and buy is their margin, which must cover costs of storage and furnish their remuneration. The excess over storage costs is a payment for specialized skill in knowing when to buy and when to sell and perhaps also for bearing risk.

This model takes for granted that there is a meaningful distinction between speculative and other transactions, that one can speak of what the price would have been in the absence of speculation. This is a point that raises many difficulties and requires careful examination in any full analysis of speculation.[2]) We can, however, evade it for our purposes by narrowing the question under discussion. Consider any market in operation. Suppose that an additional set of transactions are made in that market by an additional group of people whom we shall call "speculators" or "new speculators". We shall then deal only with the question whether this additional set of transactions

[2]) See, for example, the comments by W. J. Baumol, "Speculation, Profitability, and Stability," *Review of Economics and Statistics*, August, 1957, pp. 263—71.

increases the fluctuations in price and, if it increases them, whether it involves an economic loss or confers a gain. By dealing in this way with a change in the amount of speculation, we can avoid the troublesome intellectual problem of defining zero speculation without any essential loss in generality. We shall make one further assumption to evade a troublesome problem: namely, that the activities of speculators do not affect the quantities demanded and supplied by other participants in the market at each current price. This implies that there is a well-defined price that will clear the market at each point in the absence of speculation and that this price is not affected by speculation.

With these assumptions, it is clear that, if carrying costs are neglected, our model implies that speculators gain if they reduce inter-temporal differences in price, and lose if they widen such differences. Speculators can fill in the troughs of price movements only by buying net when prices would otherwise be low; they can flatten out the peaks only by selling net when prices would otherwise be high; unless they carry this so far as to reverse peaks and troughs, they gain by the difference. Conversely, speculators can make fluctuations wider (in the same direction) only by selling net when prices would otherwise be low and buying net when prices would otherwise be high. But this means that they sell at a lower price than they buy and so make losses. Our model therefore implicitly defines stabilizing speculation as speculation yielding gains (carrying costs aside) and destabilizing speculation as speculation yielding losses. The circumstances, if any, under which this will not be true deserve extensive examination in a full analysis of speculation but can be accepted for our more limited purposes, which is simply to show that destabilizing speculation need not involve economic loss, not that it cannot do so.[3])

[3]) See *Ibid.* for one such fuller examination. It will be clear that our assumptions rule out the main case there considered.

Baumol also considers a special case corresponding to our assumptions (pp. 269—70). His own conclusion is ambiguous but only because in judging the profitability of the speculation he does not require it to be carried through to completion, in the sense that the speculators end up in their initial position with respect to the holdings of the speculative commodity.

One reason why actual speculation might not conform to the model described in the preceding three paragraphs is *avoidable* ignorance. By no means all actions that are mistakes when viewed *ex post* fall into this category. If I wager even money that a coin will come up tails and it comes up heads, I clearly have made a mistake *ex post,* in the sense that I shall wish that I had chosen heads. If, in addition, I discover by an examination of the coin that it has heads on both sides or in some other way is biassed toward heads, and if I could have made this examination before the wager, then I have also made a mistake *ex ante.* On the other hand, if such additional examination gives me no more reason than I had before to question my belief that the coin is fair, then my initial choice of tails may be bad luck but cannot be described as a mistake. The distinction between the two cases is, in principle, whether I would have acted differently in advance of the actual toss if I had had the knowledge I gained after the toss except for the actual outcome itself, i.e., if I had had the knowledge that it would have been possible for me to have had before the toss. In the same way, the mere fact that speculators make losses over a particular period and in fact destabilize prices for that period is no evidence either that the losses could have been avoided given the general state of knowledge when the speculation was entered into or that speculation is on balance destabilizing in any more fundamental sense.

If destabilizing speculation does arise from avoidable ignorance, it must be granted immediately that there is an economic loss. The loss is borne primarily by the speculators, though if the operation is sufficiently large, second order effects on others may not be negligible in the aggregate. It may be noted in passing that insofar as this case justifies any action by government, it justifies solely the distribution of knowledge. Suppose private speculation is destabilizing because ignorant speculators behave against their own interests, but speculation by government officials trying to achieve the same end as private speculators would be stabilizing because of greater knowledge. The appropriate solution is then for the government officials to make their knowledge available either by providing the information on which their price forecasts rest or by making and publishing

the price forecasts themselves. If these are more accurate, on the whole, than the forecasts private speculators would otherwise use, private speculators have a strong incentive to act in accordance with them and in the process will produce the same results as government speculation in accordance with the same forecasts. If the forecasts are not more accurate, they will tend to be disregarded and no great harm will be done.[4])

To see how destabilizing speculation can arise without avoidable ignorance, let us start with a commodity market which is in operation. Suppose that there exists independent gambling establishments in which all gambling takes the form of betting on the future price of the commodity in question — say rubber. The people who bet on the price of rubber in the hypothetical gambling establishment do not buy or sell rubber, and neither do the people who run the establishment. Their operations therefore have no direct effect on the price of rubber; the rubber market simply takes the place of the roulette wheel at Monte Carlo.[5]) We may suppose the proprietors of an establishment to operate solely as brokers, engaging in no gambling themselves but being paid a fee for providing facilities and bringing together people willing to take opposite sides of a common wager. And we suppose throughout that the people engaging in the gambling do so deliberately and are reasonably well informed: they like to gamble and are willing to pay a price to do so. Let us put to one side any moral objections to gambling, and suppose that the gambling services are provided under competitive conditions. The proprietors of the gambling house are then devoting economic resources to producing services to satisfy the wants of consumers, who are willingly buying the services and paying a price equal to the cost of the alternative services that could have been obtained with the same resources. Clearly there is economic

[4]) One case in which publication of forecasts or the equivalent might be especially called for is if the authorities feel it necessary to suppress some relevant information for security reasons. They might be able to offset the effects of such suppression on the judgments of traders by issuing price forecasts.

[5]) There could be an indirect effect if, for example, information about the odds ruling in the gambling transactions altered the expectations about future prices of the people trading on the spot market and so changed amounts diverted to stocks.

gain rather than loss through the operation of the gambling house.[6])

Of course, there may in fact be no demand for this service at a price sufficient to call it forth. Whether there is depends on the preferences of the public for gambling of various types, the kind of gambling provided by the rubber market — that is, the probability distribution of the price of rubber — the alternative sources of gambling services, their cost and character, and so on. The willingness of people to buy lottery tickets at less than their actuarial value even though they know full well the probabilities of prizes of various size is sufficient evidence that people are willing to pay a price to bear at least certain kinds of risks, to be subjected to increased uncertainty. In any event, our concern is not with the likelihood that gambling establishments of the kind described would be profitable but only with the consequences if they were.

Consider an individual who wants to bet that the price of rubber will be higher a month from now than it is now. He can place such a bet in the gambling establishment at some odds and subject to paying a commission to the proprietors. An alternative way in which he can subject himself to the same uncertainty is to buy rubber in the market, store it for a month, and than sell it: he can accumulate positive stocks. The cost in this case is the cost of storage over the month. Similarly an individual who wants to bet that the price will fall can accomplish the same objective by selling rubber now, borrowing the physical commodity in order to make delivery currently: he can accumulate negative stocks. He may be paid for doing so, because he saves someone storage costs. Presumably however, the amount he is paid will be less than the storage costs, the difference being the fee for lending the commodity. And if the loan requires dipping into stocks needed, say, to facilitate production, storage costs may be, as it were, negative and he

[6]) It will be noted that a pure futures market is very close to such a gambling establishment. A transaction on a futures market does not by itself have any effect on the spot market. It affects current price only to the extent that the price established leads to operations on the spot market and thereby to a change in the size of stocks carried over. This is analogous to the indirect effect described in the preceding footnote.

may have to pay to borrow the goods. (Remember that we are considering the effect of the actions of an additional group of people. Their holding negative stocks simply means that total stocks are less than they would otherwise be). Suppose individuals find this alternative way of gambling cheaper. The gambling establishments will then disappear and the gambling services be provided by the rubber market.

If purchases and sales just offset, there is no effect on current price and the net costs are the various commissions paid to transact the business. The market dealers have taken the business of providing gambling services away from the gambling institutions proper. But purchases and sales need not just offset one another — indeed, the lack of necessity for them to do so may be one of the advantages of operating through the market, though a similar possibility could be provided by the gambling establishments if their proprietors "made book" rather than simply acted as brokers. If purchases and sales do not offset one another, the price of the commodity is affected.

We have now combined the two activities: gambling on the price of rubber, and the rubber market proper. Given competitive conditions, this combination will occur only if it is a cheaper way to provide gambling services and so in this respect represents increased efficiency in the use of resources. If the total expenditures of the gamblers on gambling services exceeds the commissions involved, this is equivalent to saying that, viewed as a body of speculators, they engage in destabilizing speculation. But their losses are someone's gain. In the first instance, they will be the gain of the initial participants in the rubber market. Operating on the rubber market has now become a more attractive business since one can now engage in joint production, producing gambling services as well as trading services. The result will be to attract more people into the activity. Temporary gains will be competed away and the trading margin proper reduced, so raising the average net price of rubber to the producer. But this in turn will stimulate output and so reduce the average net price of rubber to the consumer. The provision of gambling services is now being rendered jointly by the producers of rubber, the middle men, and the consumers of rubber. In return for

wider fluctuations in price — which are required to provide the gamblers or speculators with the uncertainty they want to bear — the producer gets a higher average price and the consumer pays a lower average price.

Any individual producer or consumer who disliked the wider fluctuation of prices could insure himself against it. But, given our assumptions, it cannot be that producers and consumers would be willing to pay more on the average than the difference between old and new average prices to insure themselves against the wider fluctuations. For this would contradict the initial assumption that there was a demand for the services of the gambling establishments at a positive price. The people who were willing to make bets on the price of rubber — willing to assume risks — would then have found that they were paid, instead of having to pay, for doing so. Instead of the market being supplemented by gambling institutions, it would have been supplemented by insurance companies, insuring people against the fluctuations in prices.

I grant readily that this picture of a world in which increasing fluctuations in the prices of commodities is a service that commands a positive price is hard to accept as a valid description of the actual world; not so much because people are not willing to pay for gambling — they clearly are — but because there seem to be so many cheaper ways of producing the gambles that people want to buy, though it must be noted that some of these are illegal in many countries. However, this is the picture that is implicit in the acceptance of the empirical generalization that destabilizing speculation often occurs in practice, except for such destabilizing speculation as is attributable to avoidable ignorance, or as may be consistent with deviations from our initial assumptions.

Whether particular services command a positive or negative price — are consumption services or productive services — is not determined by physical or technical considerations alone; it depends also on the tastes and preferences and the capacities and opportunities, of the community at large. Painting a fence is generally regarded as a productive service that must be paid for, as an activity yielding disutility, and so the price of painting

a fence is generally negative; Tom Sawyer was able to reverse this attitude and to make it an activity yielding utility; he was able to charge a positive price for the privilege of painting a fence. This is the essential issue involved in judging speculation. Is bearing uncertainty a service that must be paid for? Or a privilege for which people are willing to pay? Is speculation the rendering of a productive service that commands a reward? Or is it a means of gaining utility on which people spend part of their income? If it turns out to be the second rather than the first, is this any reason for regarding it as involving economic loss? Does not the tendency to do so simply reflect the preconceptions of the academic?

HAROLD T. DAVIS

The Influence of Interest Rates on Time Series of Price

1. *Introduction.*

In a paper that has become classic [1]) Harold Hotelling discusses the role of differential equations in the application of mathematics to the problems of the economist. He points out both the analytical advantages in their use and the statistical dangers which appear when they are applied. He admires the superb formulation of the astronomer with his exact equations and his accurate determination of empirical constants. This he contrasts with the less satisfactory situation of the economist, who has few rigorous laws to guide him and who must rely upon correlations for the computation of his parameters. But Hotelling points out that the astronomer has had the great advantage of a "tyrannical sun" which dominates the motion of the planets, while the economist must study a system in which the influence of any single factor is submerged in a complex of factors, none of which is dominant.

A problem in point is that of the role of the rate of interest in the dynamics of economic time series. This factor has been studied since the time of Adam Smith and David Ricardo. Volumes have been written about its influence. Irving Fisher devoted a considerable part of his scientific career to its study. The names of Walras, Pareto, Keynes, Snyder, Cassel, and many others appear in the bibliography of the subject. But in spite of this vast study, it is a curious matter to observe that one still finds it very difficult to measure with any degree of exactness the influence upon the business cycle of a change, such, for example, as that of the rediscount rate made by the board of the Federal Reserve System.

[1]) "Differential Equations Subject to Error, and Population Estimates," *Journ. Amer. Statistical Assn.*, Vol. 22 (1927), pp. 283—414.

One reason for this difficulty is to be found in the observation that this factor seems to exert its influence implicitly through a utility function, rather than as an explicit variable. One knows from abundant experience that an individual under the duress of severe financial need will agree to the payment of usurous rates of interest, but if he is in the possession of blue-chip collateral, then he is quite willing to argue the rate with his banker. What is true for the individual is also true for business. This is readily observed from the fluctuations in the interest rates on 4 to 6 months commercial paper, which has varied in this century from 7.37 % in 1920 to less than one per cent in recent years. Thus the rate of interest is observed to be a function of the individual's or the businessman's money-utility.

For this reason it has seemed to the author that some light might be thrown upon the problem if the effect of the rate of interest upon economic time series was approached strictly from the point of view of the utility function. Fortunately for this investigation a very promising beginning was made by G. Tintner in 1938,[2]) who introduced the interest rate into the familiar problem of the maximization of utility in a price-commodity space, subject to the restraint of the budget equation. A problem proposed by him, but not solved, was the investigation of second order conditions to establish criteria for the existence of a maximum for the utility function thus extended to include the rate of interest. The model suggested for such an investigation was that of Hotelling published in 1935.[3])

In the present paper the author proposes to explore the further consequences of Tintner's analysis and to integrate it with a dynamical theory of prices introduced by him in 1941.[4])

2. *A Dynamical Theory of Prices Which Includes the Effect of Interest*

We shall give a brief summary of the principal result obtained

[2]) "The Maximization of Utility over Time," *Econometrica*, Vol. 6 (1938) 154—158.

[3]) "Demand Functions with Limited Budgets," *Econometrica*, Vol. 3 (1935) pp. 71 ff.

[4]) *Analysis of Economic Time Series*, (Bloomington, Indiana, 1941), pp. 370—385.

by Tintner. Limiting himself to three quantities: x, y, and z, he denoted the amount of each consumed at time t by $x(t)$, $y(t)$, and $z(t)$ and the corresponding prices by $p(t)$, $q(t)$, and $r(t)$ respectively. He then introduced a utility function, which we shall denote by the symbol U, and a *force of interest* $\rho(t)$. The utility was then maximized subject to the budget restraint

$$E = x(t)p(t) + y(t)q(t) + z(t)r(t). \tag{2.1}$$

The equilibrium equations then assume the form

$$\frac{U'_x}{p(t)} = \frac{U'_y}{q(t)} = \frac{U'_z}{r(t)} = \lambda \exp\left[-\int_0^t \rho(s)\,ds\right] \tag{2.2}$$

where λ is the marginal utility of money at time $t = 0$. The novel element introduced into the familiar problem is the function involving the force of interest.

We shall now introduce this result into a dynamical theory of prices which the author proposed some years ago. The arguments in support of the theory have been given elsewhere and will not be repeated here. Principal support for the theory is found in the empirical evidence of its effectiveness in describing observed phenomena associated with the movement of prices, in particular, the behavior of stock prices, including the bull market of 1929, the subsequent depression, and the activity of the series in the period of recovery. But the introduction of the factor of interest into the theory as proposed here is not for the purpose of giving further support to the theory itself; it is rather to see what the effect would be when such an introduction is made. The consequences seem to have some interest, at least from the mathematical point of view, whatever one may think of their actual economic significance. And as such they are offered to the reader.

The dynamical theory of prices is based upon the following integral over some time interval: $0 \leqq t \leqq T$:

$$J = \int_0^T (U - kA - mB)dt, \tag{2.3}$$

where U is the utility function, A is a measure of the *erratic element* in the economic system, and B is a measure of the factor of supply and demand. The quantities k and m are constant

parameters of positive sign. A realistic definition of A was found to be the sum of the products of the time-rates of change of goods consumed by the time-rates of change of their prices. Thus, in the notation used above for the three-commodity system of Tintner, we write:

$$A = x'(t)p'(t) + y'(t)q'(t) + z'(t)r'(t). \tag{2.4}$$

Although this bilinear form, considered only mathematically, may have either sign, the economic argument shows that it is intrinsically positive. Hence its introduction into the integral (2.3) reduces the value of J.

The quantity B is defined to be half the sum of the squares of the differences between the quantities of good consumed and the quantities produced. If we denote by $u(t)$, $v(t)$, and $w(t)$ the quantities of x, y, and z produced, we write

$$B = \tfrac{1}{2}[(x-u)^2 + (y-v)^2 + (z-w)^2], \tag{2.5}$$

the factor $\frac{1}{2}$ being introduced only for convenience.

Although there is a tendency toward equilibrium between x, y, and z and u, v, and w, their differences may have either sign. If we define u, v, and w as current inventories at time t, and x, y, and z as the amounts currently demanded, then, in times of depressed business, $(x-u)$, $(y-v)$, and $(z-w)$ are often negative; but when business is active, as in recent years, the contrary is true. The value of B, however, is never negative and when this factor is introduced into (2.3) the value of J in general is reduced.

The primary assumption is then made that prices are determined in such a manner that the integral J is maximized over time. This leads us to the following three equations:

$$\begin{aligned} U'_x - kA'_x - mB'_x &= -k\frac{d}{dt}A'_{x'}, \\ U'_y - kA'_y - mB'_y &= -k\frac{d}{dt}A'_{y'}, \\ U'_z - kA'_z - mB'_z &= -k\frac{d}{dt}A'_{z'}. \end{aligned} \tag{2.6}$$

If we now introduce the values for U'_x, U'_y, and U'_z from (2.2) and the values for the derivatives of A and B from (2.4) and (2.5), we obtain the following equations: for the determination of the prices $p(t)$, $q(t)$, and $r(t)$:

$$(2.7)\qquad \begin{aligned} k\frac{d^2p}{dt^2} + \lambda \exp\left[-\int_0^t \rho(s)ds\right]p &= m(x-u), \\ k\frac{d^2q}{dt^2} + \lambda \exp\left[-\int_0^t \rho(s)ds\right]q &= m(y-v), \\ k\frac{d^2r}{dt^2} + \lambda \exp\left[-\int_0^t \rho(s)ds\right]r &= m(z-w). \end{aligned}$$

Without essential loss of generality we can assume that $k = 1$. If we replace the right-hand members of (2.7) by the symbols $E(t)$, $F(t)$, and $G(t)$, and if we use the abbreviation: $R(t) = \exp\,[-\int_0^t \rho(s)ds]$, then the equations can be written:

$$(2.8)\qquad \begin{aligned} \frac{d^2p}{dt^2} + \lambda R(t)p(t) &= E(t), \\ \frac{d^2q}{dt^2} + \lambda R(t)q(t) &= F(t), \\ \frac{d^2r}{dt^2} + \lambda R(t)r(t) &= G(t). \end{aligned}$$

It is the purpose of this paper to explore the consequences of this system of equations and to observe the difference in the behavior of prices between the two situations: (1) when $R(t)$ is present in the system; and (2) when $R(t)$ is replaced by a constant.

It might at first sight appear that the problem as it is formulated in equations (2.8) is oversimplified, since there is no explicit intercorrelation of the three prices as is usually the case in real markets and particularly for commodities which are competing. Generality can be restored to the system, however, by adjoining to (2.8) a functional connecting x, y, and z and their marginal rates of substitution. The obvious complexities which are introduced will be avoided in this paper by assuming independence between the variables. The general problem is thus reduced essentially to the case of a single market.

3. *Behavior of Prices Under a Constant Positive Force of Interest — Integration of the Differential Equation.*

We shall begin with a simple market involving one commodity and its price as defined by the first equation in system (2.8), that is to say, with the equation:

$$\frac{d^2p}{dt^2} + \lambda R(t)p(t) = E(t). \tag{3.1}$$

Let us assume that the force of interest is a positive constant and equal to ρ, from which we get $R(t) = e^{-\rho t}$. Equation (3.1) then assumes the form

$$\frac{d^2p}{dt^2} + \lambda e^{-\rho t}p(t) = E(t). \tag{3.2}$$

The first problem suggested by this equation is to find the behavior of the price when $E(t)$ is zero, that is to say, when the supply factor equals the demand factor. We are thus led to the solution of the equation

$$\frac{d^2P}{dt^2} + \lambda e^{-\rho t} P(t) = 0, \tag{3.3}$$

where $P(t)$ is used to denote the equilibrium price.

This equation can be integrated in terms of Bessel functions. To accomplish this we assume a new independent variable

$$z = \frac{2\sqrt{\lambda}}{\rho}e^{-\frac{1}{2}\rho t}, \tag{3.4}$$

in terms of which we compute

$$\frac{dP}{dt} = -\tfrac{1}{2}\rho z\frac{dP}{dz}, \quad \frac{d^2P}{dt^2} = \tfrac{1}{4}\rho^2 z\left(z\frac{d^2P}{dz^2} + \frac{dP}{dz}\right).$$

When these derivatives are substituted in (3.3) and obvious simplifications made, the following equation is obtained:

$$z\frac{d^2P}{dz^2} + \frac{dP}{dz} + zP = 0, \tag{3.5}$$

which is recognized as the simplest case of Bessel's differential

equation. Its general solution is the function

$$P = CJ_0(z) + DY_0(z), \tag{3.6}$$

where $J_0(z)$ and $Y_0(z)$ are respectively the Bessel functions of first and second kind of order zero.

The general solution of equation (3.3) thus assumes the form:

$$P(t) = CJ_0\left(\frac{2\sqrt{\lambda}}{\rho}e^{-\frac{1}{2}\rho t}\right) + DY_0\left(\frac{2\sqrt{\lambda}}{\rho}e^{-\frac{1}{2}\rho t}\right). \tag{3.7}$$

In order to interpret this equation, we first observe that as $t \to \infty$, we have

$$\lim_{t\to\infty} P(t) = CJ_0(0) + DY_0(0).$$

But since $J_0(0) = 1$ and $Y_0(0) = -\infty$ we must set $D = 0$ in order to obtain a realistic description of the equilibrium price.

We are thus led to a consideration of the function

$$P(t) = CJ_0(Ke^{-\frac{1}{2}\rho t}), \tag{3.8}$$

where we use the abbreviation: $K = 2\sqrt{\lambda}/\rho$.

As is well known the function $J_0(z)$ has the following asymptotic approximation provided z is sufficiently large:

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \tfrac{1}{4}\pi\right). \tag{3.9}$$

Substituting $Ke^{-\frac{1}{2}\rho t}$ for z in this formula, we obtain the following asymptotic value for $P(t)$:

$$P(t) \sim C\sqrt{\frac{2}{\pi K}}\, e^{\frac{1}{4}\rho t} \cos\left(K e^{-\frac{1}{2}\rho t} - \tfrac{1}{4}\pi\right), \tag{3.10}$$

which holds only when the argument is large.

But if we assume that $t = 0$ is the origin of our time-scale, then the largest value that z can have is K. Since $K \to \infty$ as $\rho \to 0$, it is clear that large values of the argument are to be obtained only when ρ is small.

We can now formulate a few of the characteristics of $P(t)$ as follows:

(a) If K is sufficiently large, that is to say, if ρ is small, then $P(t)$ is an oscillating function in the neighborhood of the origin.

(b) As t increases the number of zeros within any fixed interval decreases as t increases and this number ultimately reduces to zero.

(c) The relative maxima of $P(t)$ increase as t increases.

(d) The distance between relative maxima increases as t increases.

(e) $P(t)$ finally approaches C asymptotically from below.

4. *Behavior of Prices When the Force of Interest is Very Small.*

A question of considerable analytical interest is invoked when ρ is zero or very small. If ρ is actually zero, then equation (3.3) reduces to the following:

$$\frac{d^2P}{dt^2} + \lambda P(t) = 0, \tag{4.1}$$

which has the solution:

$$P(t) = a \sin (\sqrt{\lambda}\, t + b). \tag{4.2}$$

But is it very interesting to observe that $P(t)$ as defined by (3.8) does not reduce to $P(t)$ as given by (4.2) when $\rho \to 0$, since $K \to \infty$, and the function approaches zero for all values of t.

It is important, however, to have a solution of equation (3.3) when ρ is small, that is to say, a solution intermediate between (4.2) and (3.8). Such a solution can be developed formally as a power series in ρ, that is, in the form:

$$P_0(t) = a \sin (\sqrt{\lambda}\, t + b) + \rho\phi_1(t) + \rho^2\phi_2(t) + \ldots. \tag{4.3}$$

Since *secular terms*, that is to say, terms of the form $t^n \sin \sqrt{\lambda}\, t$ and $t^n \cos \sqrt{\lambda}\, t$, are thus introduced, one has reservations about the convergence of the series. But it is not difficult to show that for a limited range of t and for ρ sufficiently small the difference between $P(t)$ as defined by (3.8) and the first two terms of $P_0(t)$ can be made arbitrarily small.

After some effort the value of $\phi_1(t)$ was finally obtained and $P_0(t)$ written in the following somewhat more convenient form:

$$(4.4)\qquad \begin{aligned} P_0(t) &= a_0\left[\cos\alpha t + \frac{\rho}{4\alpha}(-\sin\alpha t + \alpha t\cos\alpha t + \alpha t^2\sin\alpha t) + \ldots\right] \\ &\quad + a_1\left[\sin\alpha t + \frac{\rho}{4\alpha}(t\sin\alpha t - \alpha t^2\cos\alpha t) + \ldots\right], \end{aligned}$$

where a_0 and a_1 are arbitrary constants and $\alpha^2 = \lambda$.

In order to compare $P(t)$ as defined by (3.8) with $P_0(t)$, we equate both functions and their first derivatives at $t = 0$. We thus find

$$(4.5)\qquad a_0 = CJ_0(K), \quad a_1 = CJ_1(K),$$

where $J_1(z)$ is the Bessel function of first order.

The magnitude of the error $|P(t) - P_0(t)|$ is shown by the following numerical approximations:

Setting $C = 100$, $\lambda = 1$, $\rho = 0.1$, $t = 2$, we compute:

$$P(t) = 100\, J_0(18.09675) = 0.48167, \ P_0(t) = 0.49995,$$

which is quite satisfactory, considering the comparatively large value of ρ, namely, a force of interest equal to 10 %.

5. *Behavior of Prices When the Force of Interest is Negative.*

There is nothing inherent in the theory of Tintner to prevent us from assuming that the force of interest in $R(t)$ is negative. That is to say, we can write: $R(t) = e^{\rho t}$, $\rho > 0$, and thus replace equation (3.3) by the following:

$$(5.1)\qquad \frac{d^2\pi}{dt^2} + \lambda e^{\rho t}\pi(t) = 0,$$

where $\pi(t)$ denotes the price at time t.

The meaning to be attached to this assumption is readily seen from Tintner's fundamental equation, namely

$$(5.2)\qquad I_j + s_{j-1}(1 + i_{j-1}) = E_j + s_j, \qquad j = 1, 2, \ldots, n,$$

$s_n = 0$, where I_j is the expected money income, E_j the expected expenditure, and s_j the expected savings at time j, and s_{j-1} the savings planned for time $j - 1$. These savings are accumulated at an anticipated rate of interest i_{j-1}. But as one well knows savings are not always invested at a profit, in which case i_{j-1} is

negative. The theory leads to the interesting conclusion that the anticipated marginal utility of money at time $j-1$ may be written

$$\lambda = \lambda_1/[(1+i_1)(1+i_2)\dots(1+i_{j-1})], \tag{5.3}$$

where λ_1 is the observed marginal utility at the origin of the time sequence.

Assuming that the rates of interest are positive over a given period of time, then (5.3) can be replaced by the marginal utility: $\lambda = \lambda_1 e^{-\rho t}$; but in the contrary case, if the rates of interest are negative, then the utility becomes: $\lambda = \lambda_1 e^{\rho t}$.

Before considering the actual economic consequences of this assumption, let us first consider the solution of equation (5.1). Observing that $J_0(-z) = J_0(z)$, we obtain the desired solution by replacing ρ by $-\rho$ in equation (3.8). We thus have

$$\pi(t) = CJ_0(K e^{\frac{1}{2}\rho t}), \tag{5.4}$$

where $K = 2\sqrt{\lambda}/\rho$ and C is an arbitrary constant.

From the properties of the function $J_0(z)$ we readily obtain the following characteristics of $\pi(t)$:

(a) $\pi(t)$ declines in an oscillatory manner from an initial value $\pi(t_0)$ and the frequency of its oscillations increases as t increases.

(b) The maxima of $\pi(t)$ are asymptotic to the function: $C\sqrt{2/\pi K}\, e^{-\frac{1}{4}\rho t}$.

(c) $\pi(t)$ approaches zero as $t \to \infty$.

6. *Graphical Representation of the Functions.*

Before considering the economic realities of the theory which we have described, let us examine the graphical representation of the functions for a few special values of the parameters.

Case 1. If $\rho > 0$, then we have: $P(t) = CJ_0(K e^{-\frac{1}{2}\rho t})$, where $K = 2\sqrt{\lambda}/\rho$. Let us introduce arbitrarily the values: $C = 10$, $\lambda = 0.09$, $\rho = 0.04$, whence $K = 15$. This function we shall consider over the range: $0 \leqq t \leqq 150$, that is to say, we shall evaluate $J_0(z)$ between $z = 15$ and $z = 0.75$, where $z = 15\, e^{-0.02t}$. Values of $P(t)$ over this range are contained in the following table:

t	z	$P(t)$	t	z	$P(t)$
0	15.0	−0.14	60	4.5	−3.21
10	12.3	−1.11	70	3.8	−4.01
20	10.0	−2.46	80	3.0	−2.60
30	8.3	1.09	100	2.0	2.07
40	6.8	2.89	125	1.23	6.53
50	5.5	0.10	150	0.75	8.64

The maxima, minima, and zero values of $P(t)$ over the given range are contained in the following table:

t	z	$P(t)$	t	z	$P(t)$
0+	14.93	0	38.0	7.02	3.00
6.1	13.32	2.18	50.0	5.52	0
12.0	11.79	0	68.5	3.83	−4.03
19.3	10.17	−2.50	91.5	2.40	0
27.5	8.65	0			

From these values the graph of $P(t)$ is now constructed between $t = 0$ and $t = 150$ as shown in Figure 1. In this same figure

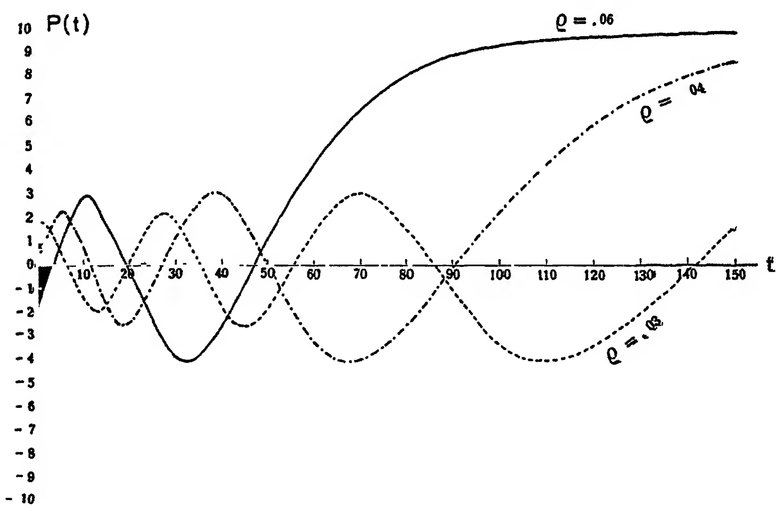

Fig. 1.

there are shown also the graphs corresponding to $\rho = 0.03$ and $\rho = 0.06$, which were similarly computed.

We observe from these figures that $P(t)$ for positive values of ρ oscillates about the base line, but ultimately approaches asymptotically the line $P = 10$. We also see that this value is approached more rapidly the larger ρ is taken and that at the same time the number of oscillations diminishes. It is also worthy of note that as ρ approaches zero the curve tends towards a sine curve, but as the number of oscillations increases the amplitude of the maximum values diminishes.

Case 2. If $\rho < 0$, then we are concerned with $P(t) = CJ_0(K\,e^{\frac{1}{2}\rho t})$. Introducing the arbitrary values: $C = 10$, $\sqrt{\lambda} = 0.02$, $\rho = 0.04$, we consider this function over the range: $0 \leqq t \leqq 150$. In this case $P(t)$ decreases in a sinusoidal pattern from an arbitrary initial value of $10\ J_0(1) = 7.65$ to a limiting value of 0. This curve is shown in Figure 2, in which is also exhibited the graph corresponding to $\rho = 0.06$.

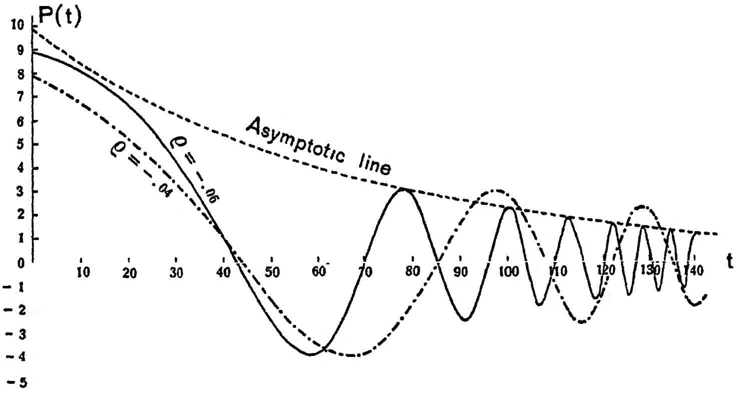

Fig. 2.

We now observe that the number of oscillations increases as ρ is increased and that the amplitudes of the maximum values diminish more rapidly. The curve which is asymptotic to these maxima, for $\rho = 0.06$, is also shown in the figure. Its equation is readily found to be the following:

$$y = C\sqrt{\frac{2}{\pi K}}\,e^{-\frac{1}{4}\rho t} = 9.772\,e^{-0.015t}.$$

7. *Economic Applicability of the Theory.*

It not infrequently happens that a pleasing mathematical formulation of an economic problem may actually have little application to the phenomena which is proposes to explain. Either the parameters may resist statistical estimation; or, if they can be determined, the real pattern and the mathematical pattern may be at variance with one another. It is thus a matter of some importance to put any set of equations to the test of reality.

In the present case we can find at least one period where the present theory appears to have applicability, namely, the years between 1917 and 1935. Within these dates the movements of the business cycle were essentially those of a free economy, which had been stimulated by the war economy that had preceded it and which had not yet been affected by the intrusion of government spending, high income taxes, and other regulations that characterized the subsequent period. Large variations both in the index of prices and in the rate of interest are to be observed. The rise and fall of stock prices, a balanced national budget, and a decline in the national debt were also characteristic features of these years.

But the reason why this period is particularly adapted to the present study is found in the observation that the principal criterion of Tintner's theory, the final disappearance of savings, is satisfied. To show this we shall assume that in 1917 a capital fund of \$ 100,000 is invested at the current rate of interest, i_j. While the rate of interest may have numerous definitions, it will be convenient to assume this rate to be that of 4 to 6 months commercial paper. The income thus derived in the j-th period is I_j. From this income an expenditure is taken which we shall assume has a fixed purchasing value equivalent to \$ 4,000 in 1917. Thus $E_j = \$\ 4{,}000\ P_j$, where P_j is an appropriate price index with 1917 as base. The general price index of the Federal Reserve Bank of New York City seemed to be quite suitable for his purpose.

In order to bring Tintner's theory within the scope of this paper it is convenient to introduce an average rate of interest, i_A, in terms of which we can then write equation (5.2) in the

following equivalent form:

$$I_j + (1 + \rho_j)s_{j-1} = E_j + s_j, \tag{7.1}$$

where we have: $I_j = C\, i_j + s_{j-1} i_A$, C = the initial capital, and $\rho_j = i_j - i_A$. As we have assumed above, C = \$ 100,000. The average rate of interest, computed from the formula: $(1 + i_A)^n = (1 + i_1)(1 + i_2) \ldots (1 + i_n)$, was found to be 4.16 %. With these values the following schedule was then computed:

Year	Capital fund	Price Index 1913=1.00	Interest rate	ρ_j	I_j	$(1+\rho_j)s_{j-1}$	E_j	s_j
1917	\$100,000	1.39	4.74%	0.58%	\$4740	\$ 0	\$4000	\$ 740
1918	100,740	1.57	5.87	1.71	5901	753	4520	2133
1919	102,133	1.73	5.42	1.26	5509	2160	4960	2709
1920	102,709	1.93	7 37	3.21	7483	2796	5560	4719
1921	104,719	1.63	6.53	2 37	6726	4831	4680	6877
1922	106,877	1.58	4.43	0.27	4716	6896	4560	7052
1923	107,052	1.65	4.98	0.82	5273	7110	4760	7623
1924	107,623	1.66	3.91	−0.25	4227	7604	4760	7071
1925	107,071	1.70	4.03	−0.13	4324	7062	4880	6506
1926	106,506	1.71	4.24	0·08	4511	6511	4920	6102
1927	106,102	1.71	4.01	−0.15	4264	6093	4920	5437
1928	105,437	1.76	4.84	0.68	5066	5474	5080	5460
1929	105,460	1.79	5.78	1.62	6007	5548	5160	6396
1930	106,396	1.68	3.56	−0.60	3826	6358	4840	5344
1931	105,344	1.50	2.64	−1.52	2896	5263	4320	3805
1932	103,805	1.32	2.84	−1.32	2998	3755	3800	2953
1933	102,953	1.29	1.87	−2.29	1993	2885	3720	1158
1934	101,158	1.37	1.14	−3.02	1188	1123	3960	-1649
1935	98,351	1.45	0.91	−3.25	—	—	—	—

It will be clear both from this table and from the graphs shown in Figure 3 that there is a general tendency in this period for interest rates and prices to move together. We shall now attempt to show that the dynamical theory which we have introduced is in satisfactory agreement with the observed behavior of prices.

We first observe that the force of interest varies from a maximum of 3.21 % to a minimum of −3.25 %. Between 1917 and 1923 inclusive it has an average value of 1.45; between 1924

and 1929 its average is 0.029 and between 1930 and 1935 its average is −1.75. Corresponding to each of these periods we shall now assume the following values of ρ : $\rho_1 = 0.0145$, $\rho_2 = 0.0029$, and $\rho_3 = -0.0175$.

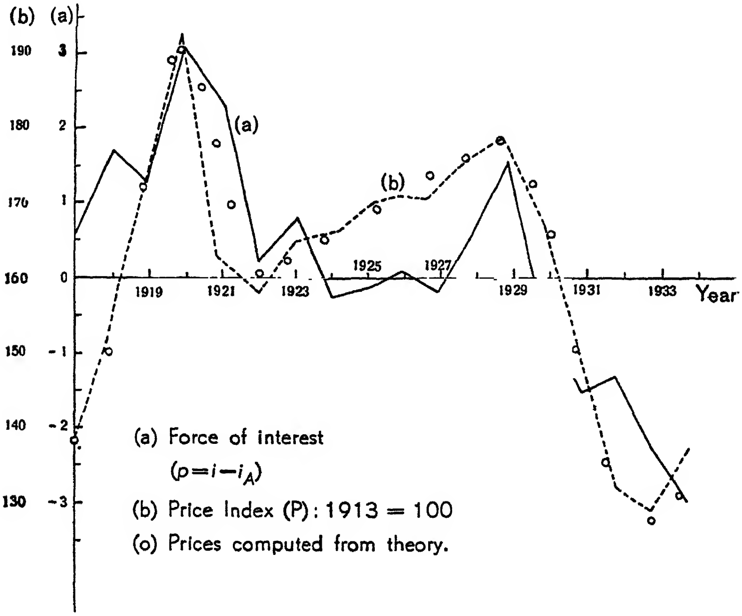

Fig. 3

If we now denote by P_1 the variation in price in the first period from an average value of 1.60, then the theory states that this variation can be represented approximately by the function:

$$P_1 = C_1 J_0\left(\frac{k}{\rho_1} e^{-\frac{1}{2}\rho_1 \tau}\right), \tag{7.2}$$

and similarly for the third period by

$$P_3 = C_3 J_0\left(\frac{k}{\rho_3} e^{-\frac{1}{2}\rho_3 \tau'}\right), \tag{7.3}$$

where τ and τ' are variables measuring time from an initial value of zero.

Since the marginal utility of money has presumably not changed throughout the time interval, the constant k is the

same in each function. To identify the initial value of P_1 with the observed value of the actual price index, we let $C_1 = 1$ and $k/\rho_1 = 10$. It then follows that $k/|\rho_3| = 10(\rho_1/|\rho_3|) = 8.29$.

Since both τ and τ' are variables of time, but in unspecified units, it is clear that if time (t) is measured in some standard unit such as months, we must write: $\tau = at + b$, $\tau' = a't + b'$, where a, b, and a', b' are constants. In fitting (7.2) and (7.3) to the observed data, where t in months varies from 0 to 204, we find the following transformations are necessary: $\tau = 0.862t$, and $\tau' = 0.354t + 0.413$. Introducing these values respectively into (7.2) and (7.3), and determining $C_2 = 2$ from the observed data, we obtain the following functions as approximations to the series of price variations within the two periods:

$$(7.4) \qquad P_1 = J_0(10\, e^{-0.00624\, t}), \quad P_3 = 2J_0(12.53\, e^{0.0031\, t}).$$

In the second period, since ρ_2 is very small, a sine curve is fitted to the data. A satisfactory representation is obtained from the following function:

$$(7.5) \qquad P_2 = \sin\,(0.00229t - 0.137).$$

The excellence of the approximation thus achieved by (7.4) and (7.5) is readily seen from Figure 3, where the circles represent the values computed from these functions.

8. *Behavior of Prices Under a Constant Force of Interest and an Impressed Force.*

During the period which we have just analyzed in Section 7 there is to be observed another price series which does not conform at all to the pattern of general prices. We refer to the price of industrial stocks as measured by the Dow Jones averages. The great difference between the two series is readily shown, for example, in the period between 1923 and 1929 when the Dow Jones average increased from 100 to a maximum of 340, while the general price index changed but little. A similar spectacular difference is also to be observed in the period between 1950 and 1957, when stock prices, as measured by the Dow Jones averages, increased from a level of 200 to a maximum well over 500. During this period general prices showed a modest increase of approximately 15%.

The original intention of the author's theory of price dynamics was to explain, first, an observed cyclical pattern in stock prices and, second, the abnormal rises which we have just described. Considerable success, measured in terms of the statistical agreement between theory and observation, has attended this explanation. Since both of the periods have been characterized by an abnormal business activity and since the rise in stock prices appears to be a direct concomitant of it, one is led naturally to seek a mathematical explanation in the non-homogeneous equation:

$$\frac{d^2p}{dt^2} + \lambda\, e^{-\rho t} p(t) = E(t), \tag{8.1}$$

where $E(t)$ is a measure of industrial activity.

In the author's original study, the effect of the interest rate was not included. In spite of this omission the dominating influence of the force function, $E(t)$, made possible a satisfactory explanation of the explosive prices of the bull market through the phenomenon of resonance. But the long cycles observed in the movement of stock prices after the collapse of the bull market were not adequately accounted for by the restricted theory except by the introduction of certain *ad hoc* hypotheses. It seems highly probable that this behavior can be accounted for in a satisfactory manner by the introduction of the force of interest.

The most conspicuous factor that was neglected in the earlier study was the influence of the rediscount rate of the Federal Reserve Board upon the action of stock prices. This controversial matter — the actual effect of a change in the rediscount rate — has been difficult to appraise satisfactorily. But it seems quite possible that equation (8.1) may provide a new and more satisfactory approach to the problem. This can be achieved by making a comparison between the movement of prices for a given observed $E(t)$ when ρ is set equal to zero, and when it is given a positive or a negative value. Unfortunately the scope of the present paper precludes an adequate statistical study of this engaging problem, but at least its mathematical formulation is possible.

Let us begin by introducing the transformation of the independent variable t to the variable z by means of equation (3.4). We thus replace (8.1) by the following:

$$z^2 \frac{d^2 p}{dz^2} + z \frac{dp}{dz} + z^2 p = \frac{4}{\rho^2} E_1(z), \tag{8.2}$$

where we use the abbreviation:

$$E_1(z) = E\left[\frac{2}{\rho} \log\left(\frac{K}{z}\right)\right], \quad K = 2\sqrt{\lambda}/\rho. \tag{8.3}$$

The solution of equation (8.2) can be written in the form:

$$\begin{aligned} p(z) &= AJ_0(z) + BY_0(z) \\ &\quad + \frac{8\pi}{\rho^2}\int^z [J_0(s)Y_0(z) - J_0(z)Y_0(s)] \frac{E_1(s)}{s}\, ds, \end{aligned} \tag{8.4}$$

where $J_0(z)$ and $Y_0(z)$ are respectively Bessel functions of first and second kind of order zero. The quantities A and B are arbitrary constants.

If $\rho = 0$, then (8.4) is to be replaced by the following:

$$p(t) = A \sin \alpha t + B \cos \alpha t + \frac{1}{\alpha}\int_0^t \sin \alpha(t - s)E(s)ds, \tag{8.5}$$

where A and B are arbitrary constants and $\alpha^2 = \lambda$. As in the homogeneous case, there appears to be no obvious way by means of which we can pass from (8.4) to (8.5) by a limiting process when $\rho \to 0$.

Returning to equation (8.4), we see that it is possible to find the price which corresponds to a given impressed force, but there are obvious difficulties of integration which make the determination of price by this means impractical in general. Except for a few cases, which are unrealistic in the present problem, the evaluation of the right hand member of (8.4) must be achieved by the methods of mechanical integration.

It should be observed, however, that if an observed price is given, the problem of finding the impressed force which generates it, is much easier to solve. In this case, we can regard both (8.4) and (8.5) as Volterra integral equations, from the inversion of which the desired function is obtained. Since, moreover, the

differential equivalents of the two equations are known, the solution of the problem is readily obtained.

Thus, the solution of equation (8.1) can be written:

$$p(t) = P_0(t) + Q(t), \tag{8.6}$$

where $P_0(t)$ is the price when $E(t) = 0$, and $Q(t)$ is the part of price which is generated by the force function. If, then, we denote by $L(u)$ the operator

$$L(u) \equiv \frac{d^2 u}{dt^2} + \lambda\, e^{-\rho t} u(t), \tag{8.7}$$

equation (8.1) can be written in the form:

$$E(t) = L(P + Q) = L(P) + L(Q) = L(Q). \tag{8.8}$$

Therefore, given an observed price series, it is possible to obtain from (8.8) the market, which would lead to such a series. A few examples are given below as follows:

Case 1. $p = P + K$, where K is a constant. $E(t) = K\, e^{-\rho t}$. In this case, the market starts with a demand in excess of supply, but equality between demand and supply is attained exponentially. (Figure 4).

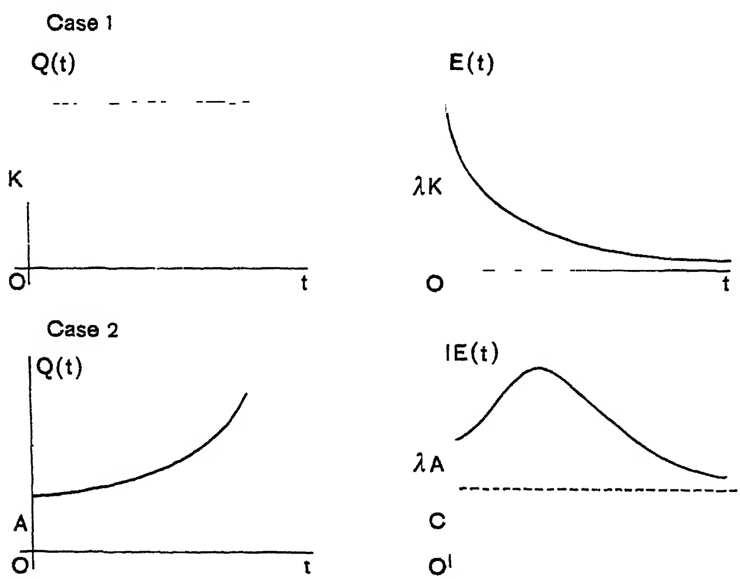

Fig. 4

Case 2. $p = P + A + Bt + Ct^2$, where A, B, and C are positive constants. $E(t) = 2C + \lambda\, e^{-\rho t}(A + Bt + Ct^2)$. In this case the market starts with a demand in excess of supply, which ultimately reaches a point of constant equilibrium in which the difference between the quantities demanded and those supplied is constant. Prices then increase according to a quadratic law and the effect of the interest rate is finally absorbed. (Figure 4).

Case 3. $p = P + A + Bt$, $A, B > 0$. $E(t) = \lambda\, e^{-\rho t}(A + Bt)$. In this case the market starts with a demand in excess of supply, which, after rising to a maximum value, decreases exponentially until equality is approximated between the two factors. Prices rise linearly and ultimately absorb the effects of the interest rate. (Figure 5).

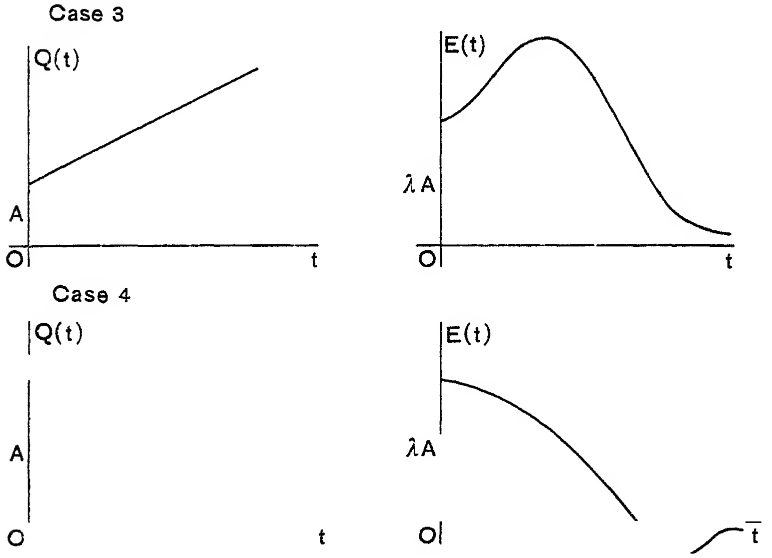

Fig. 5

Case 4. $p = P + A + Bt$, $A > 0$, $B < 0$. $E(t) = \lambda\, e^{-\rho t}$ $(A + Bt)$. As in Case 3, the market starts with demand in excess of supply, but the demand at once declines exponentially until equality is reached. Thereafter, the supply is slightly in excess of demand, but the difference diminishes asymptotically to

zero. Prices themselves decline linearly until the effect of the interest rate is finally absorbed. (Figure 5).

Case 5. $p = P + K + A \sin (kt + c)$, $K > 0$. $E(t) = -Ak^2 \sin (kt + c) + K + A \sin (kt + c)\, e^{-\rho t}$. In this case the market, which starts with a sinusoidal movement about a constant positive demand, finally damps into a pure sinusoidal movement in which there is at one time an over supply and in another excess demand.

Case 6. $(\rho = 0)$. $p = -(A/2\alpha)\, t \cos \alpha t$, $\alpha^2 = \lambda$. $E(t) = A \sin \alpha t$. In this case we have an example of the effect of resonance. A sinusoidal market, which has the same period as that of the free system, generates as explosive price. The bull market of 1929 is readily explained by this phenomenon, as has been indicated earlier in this paper.

Something akin to this situation is also observed when ρ is positive. Thus if we assume that $p = tJ_0(z)$, where z is defined by (3.4), then we find that $E(t) = \rho z J_1(z)$, where $J_1(z)$ is the Bessel function of first order. At t increases z approaches zero and $J_0(z)$ approaches unity. Hence p increases without limit. But as z approaches zero, $J_1(z)$ also approaches zero and $E(t)$ is ultimately zero. We thus have an explosive price with a force function that decreases asymptotically to zero.

PART IV

On the Edgeworth Taxation Paradox

WILLIAM VICKREY

Can Excises Lower Prices?

In the land of *ceteris paribus*, that kindergarten of economic theory where one thing changes at a time while all else remains frozen on the spot as in a game of "redlight," it is a commonplace that if a tax is placed upon an article, its price to the consumer will be raised, or will at most remain the same. This proposition, true enough when "other things" are "equal", in some sense, has quite generally been supposed to apply also in the real world where "ceteris" are notoriously reluctant to remain "paribus". To be sure, it is difficult, offhand, to produce an example from real life that can confidently be said to controvert this proposition. But the complacency with which economists had been extending their results from the partial equilibrium world of ceteris paribus to the real world of complex interrelations was, for the more sophisticated of them, rather rudely shocked when Edgeworth, in 1897, managed to produce an apparently reasonable and consistent example of a case where a tax on a certain class of railway ticket could induce a profit-maximizing monopolist to *reduce* the rate for *all* classes of tickets, inclusive of tax.[1])

However, the effect of this paradox on economic thinking has been considerably less devastating than might have been expected. The model of necessity involved considerable mathematics, and those outside the inner circle of economists versed in mathematics were prone to suspect some piece of legerdemain concealed in the works. Furthermore the example involved monopoly, and at that, a type of monopoly generally subjected to public control rather than left to maximize its profit as best it might, so that even those few economists who were familiar

[1]) F. Y. Edgeworth, "La Teoria Pura del Monopolio," *Giornale degli Economist*, Vol. 15 (October, 1897), pp. 307—320, esp. p. 317. "The Pure Theory of Taxation", *Economic Journal*, VII (June, 1897) 231; reprinted in *Papers Relating to Political Economy* (London, 1925), I, 132, 143; II, 93, 401.

with the paradox were inclined to treat the example as a curiosum, not likely to have any real life counterpart. The reaction of Seligman is perhaps typical: "The mathematics which can show that the result of a tax is to cheapen the untaxed as well as the taxed commodities will surely be a grateful boon to the perplexed and weary secretaries of the Treasury and ministers of finance throughout the world!"[2])

It remained for Harold Hotelling to show, in 1932, that the phenomenon was also possible under conditions of perfect competition; but again the rather formidable looking mathematics involved tends to scare away most economists. All that Hotelling says in non-mathematical terms about the conditions that are required, if a tax levied on one of two related commodities is to lower both their prices, is that the two commodities must compete both in production and in consumption. But obviously this is not enough of itself to produce the paradox, and we cannot say that the phenomenon is likely merely because a large number of commodities satisfy these conditions. Something more is obviously required. It is the purpose of this article to set forth with the aid of as elementary mathematics as is possible just what that something more is.

Unfortunately there seems to be no way to avoid some recourse to mathematics. But I will try to come out with something more than a mere set of equations. To simplify the discussion as much as possible, it will be assumed that we are dealing with only the two-commodity case under conditions of perfect competition. That is, we will consider only the case where the paradox can arise through the interrelations between only two commodities, and cases involving longer and more complicated chains of relations will be left aside for the moment. Also, cases requiring a greater or lesser degree of monopoly will be ignored. Moreover, it will further simplify matters to assume that the importance of the commodities considered in the budgets of individuals is so small that changes in the marginal

[2]) E. R. A. Seligman, *Shifting and Incidence of Taxation* (4th ed.; New York, 1921), p. 214n.

[3]) Harold Hotelling, "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions", *Journal of Political Economy*, XL (1932), 577—616.

utility of money, i.e., the "income effect" can be neglected and that individuals are able to devote more or less of their resources to the particular commodities considered without large changes in standard of living.

It can be shown that the existence of the Edgeworth paradox, for sufficiently small rates of tax, depends only on the second derivatives of the utility and cost functions, which correspond to the first derivatives of the demand and supply functions. Accordingly we may approximate the utility and cost functions in the neighborhood of the equilibrium point (with no taxes) by functions of the second degree only.[4])

Accordingly we may write:

$$W = a(x - \bar{x})^2 + 2b(x - \bar{x})(y - \bar{y}) + c(y - \bar{y})^2 + \bar{p}_x(x - \bar{x}) + \bar{p}_y(y - \bar{y}) + \overline{W} \tag{1}$$

for the total cost function, W being the total cost of producing amounts x and y of the two commodities, the remaining symbols being constants. If $\bar{x}$ and $\bar{y}$ are the quantities produced under equilibrium conditions with no tax, then $\overline{W}$ is the cost of producing this equilibrium output. Under competitive conditions the price realized by producers will be equal to marginal cost, so that we can put

$$\begin{aligned} p_x &= M_x = \frac{\partial W}{\partial x} = \bar{p}_x + 2a(x - \bar{x}) + 2b(y - \bar{y}) \\ p_y &= M_y = \frac{\partial W}{\partial y} = \bar{p}_y + 2b(x - \bar{x}) + 2c(y - \bar{y}) \end{aligned} \tag{2}$$

Thus the constants $\bar{p}_x$ and $\bar{p}_y$ are the equilibrium prices obtaining in the absence of tax, justifying the symbol used. It will be shown in due course that all of the constants must be positive. Indeed for an equilibrium in which sellers assume prices to be constant, the marginal cost of x must increase with increasing x, and similarly for y, so that the constants a and c are immediately seen to be positive. A third requisite for the equilibrium to be stable is that ac exceed b^2. This condition corresponds to the increasing marginal rate of substitution, or that the curves of constant cost in the x, y, plane be concave towards the origin.

[4]) For the completely general treatment the reader is referred to the Hotelling article cited above.

These conditions are necessary in order that the producer be at a point of maximum profit rather than of minimum profit or possibly at a minimax or saddle point (which would admit of increasing profits by changing output, assuming prices to remain constant.)

Similarly we may write

$$U = -A(x-\bar{x})^2 - 2B(x-\bar{x})(y-\bar{y}) - C(y-\bar{y})^2 + \bar{P}_x(x-\bar{x}) + \bar{P}_y(y-\bar{y}) + \bar{U} \tag{3}$$

for an aggregate utility function, the utility function of each individual being added in with such weight as to make the marginal utility of money unity for all persons. With total utility thus measured in terms that make the marginal utility of money one, we can equate prices to consumers with marginal utilities as follows:

$$\begin{aligned} P_x &= \frac{\partial U}{\partial x} = U_x = \bar{P}_x - 2A(x-\bar{x}) - 2B(y-\bar{y}) \\ P_y &= \frac{\partial U}{\partial y} = U_y = \bar{P}_y - 2B(x-\bar{x}) - 2C(y-\bar{y}) \end{aligned} \tag{4}$$

Again we see that $\bar{P}_x$ and $\bar{P}_y$ turn out to be the prices obtaining at the equilibrium consumption $\bar{x}$ and $\bar{y}$. The constants A and C must be positive, if the marginal utility of x is to decrease with increasing x, and similarly for y. Further, for consumers to be in stable equilibrium when confronted with the opportunity to buy at prices they consider constant, we must have AC greater than B^2. This corresponds to the diminishing marginal rate of substitution for consumers, and insures that the indifference curves in the xy plane will be concave away from the origin. These conditions are necessary in order that consumers be at a point of maximum utility rather than at a minimum or a minimax, assuming that prices are to be constant. If these conditions are not satisfied it would be possible for consumers to improve their position by moving away from the equilibrium point in a proper direction.

Now the price to the consumer will exceed the price realized by the seller by the amount of the tax, so that we have

$$p_x + t_x = P_x,\ p_y + t_y = P_y;\ \bar{p}_x = \bar{P}_x,\ \bar{p}_y = \bar{P}_y. \tag{5}$$

Solving these equations for the tax rates, and making use of (2) and (4), we have

$$(6)\quad \begin{aligned} t_x &= P_x - p_x = -2(A+a)(x-\bar{x}) - 2(B+b)(y-\bar{y}) \\ t_y &= P_y - p_y = -2(B+b)(x-\bar{x}) - 2(C+c)(y-\bar{y}). \end{aligned}$$

These equations can be solved for x and y in terms of the tax rates, giving us

$$(7)\quad x = \bar{x} - \frac{2\begin{vmatrix} t_x & B+b \\ t_y & C+c \end{vmatrix}}{4\begin{vmatrix} A+a & B+b \\ B+b & C+c \end{vmatrix}} = \bar{x} - \frac{1}{2D}[t_x(C+c) - t_y(B+b)]$$

$$(8)\quad y = \bar{y} - \frac{2\begin{vmatrix} A+a & t_x \\ B+b & t_y \end{vmatrix}}{4\begin{vmatrix} A+a & b+b \\ B+b & C+c \end{vmatrix}} = \bar{y} - \frac{1}{2D}[t_y(A+a) - t_x(B+b)]$$

$$(9)\quad \text{where } D = \begin{vmatrix} A+a & B+b \\ B+b & C+c \end{vmatrix} = (A+a)(C+c) - (B+b)^2$$

these expressions can in turn be substituted for x and y in equations (4), yielding the following expressions for the prices to consumers in terms of the taxes:

$$(10)\quad \begin{aligned} P_x &= \bar{P}_x + \frac{A}{D}\begin{vmatrix} t_x & B+b \\ t_y & C+c \end{vmatrix} + \frac{B}{D}\begin{vmatrix} A+a & t_x \\ B+b & t_y \end{vmatrix} \\ &= \bar{P}_x + \frac{t_y}{D}[A(C+c) - B(B+b)] + \frac{t_y}{D}(aB - Ab) \\ P_y &= \bar{P}_y + \frac{B}{D}\begin{vmatrix} t_x & B+b \\ t_y & C+c \end{vmatrix} + \frac{C}{D}\begin{vmatrix} A+a & t_x \\ B+b & t_y \end{vmatrix} \\ &= \bar{P}_y + \frac{t_y}{D}[C(A+a) - B(B+b)] + \frac{t_x}{D}(Bc - bC). \end{aligned}$$

Consider the effect of imposing a tax on x. If this is to diminish the prices of both x and y, we must have, from (10)

$$(11)\quad \frac{\partial P_x}{\partial t_x} = \frac{1}{D}[A(C+c) - B(B+b)] < 0; \quad \frac{\partial P_y}{\partial t_x} = \frac{1}{D}(Bc - bC) < 0$$

D, however, can be shown to be positive. For D can be written:

(12) $\quad D = (AC - B^2) + (ac - b^2) + (aC + Ac - 2Bb);$

the first two parentheses are positive by reason of the maximum conditions; in the final parenthesis we have:

(13) $$aC + Ac > 2\sqrt{aCAc} = 2\sqrt{AC}\sqrt{ac} > 2Bb,$$

so that all three parentheses are positive and thus D is positive. Hence the conditions for a tax on x to increase the prices of both x and y are:

(14) $$A(C + c) - B(B + b) = AC - B^2 + Ac - Bb < 0$$

(15) $$Bc - bC < 0.$$

To show that b and B are positive, we multiply (14) by C (which is positive), and add and subtract B^2c, getting:

(16) $$c(AC-B^2)+ACc-BCb-B^2c+B^2c < 0,$$
$$\text{or } (C+c)(AC-B^2)+B(Bc-bC) < 0.$$

In this last expression the first two parentheses are both positive, and the last parenthesis is negative from (15); thus if the whole expression is to be negative, B must be positive. If B is positive, then from (15) b must also be positive. This means that the two commodities x and y compete both in production and consumption. That is, the more of one that is consumed, the smaller will be the marginal utility of the other, and the more of one that is produced, the greater will be the marginal cost of producing the other. This is the condition verbalized by Hotelling. Combining the conditions (14) and (15) with the conditions that a, b, c, A, B, and C all be positive, and the convexity conditions

(17) $$ac - b^2 > 0; \; AC - B^2 > 0$$

we can obtain the following chain of relationships:

(18) $$\frac{a}{b} = \frac{ac-b^2}{bc} + \frac{b}{c} > \frac{b}{c} > \frac{A}{B} + \frac{AC-B^2}{Bc} > \frac{A}{B}$$
$$= \frac{AC-B^2}{BC} + \frac{B}{C} > \frac{B}{C}.$$

In this chain, the first relation is an identity, the second follows from (17) (plus the fact that all the coefficients are positive); the third follows from (14), the fourth again from (17), the

fifth is an identity, and the sixth follows from (17). (15) is needed only to show that the coefficients are actually positive, and in fact if all coefficients are positive, (15) follows immediately from the last four relations in the chain. That is, if a tax on x lowers the price of x, and the commodities compete in production and in consumption, the tax on x will also lower the price of y.

It is not enough, however, to produce the paradox, to say that the two commodities must compete in consumption and production. To examine these conditions further, we can rewrite the utility and cost functions in such a way as to split up the two commodities into two independent factors. That is, we can think of the production of the commodities as being carried on in, say, two independent stages, one stage being common to both products, and the other production process being applied to only one. For example, one can consider as two separate products ground coffee (x) and coffee in the bean (y), the growing, curing, and roasting of the coffee being one process applied to both products and the grinding a supplementary process applied only to the x. If these processes are independent in the sense that the volume of one process has no effect on the costs involved in the other, then we should be able to write $W=f(x)+g(y+\alpha x)$, where α is the number of units of x that correspond to a unit of y in the common process; in the case of ground and bean coffee, this is obviously pound for pound, or unity. In general, we find that in order for the processes to be independent, we must put

$$\alpha = \frac{b}{c} \tag{19}$$

and we can write the cost function as:

$$\begin{aligned} W= \frac{ac-b^2}{c}x^2+\left(\bar{p}_x-\frac{b}{c}\bar{p}_y-2\frac{ac-b^2}{c}\bar{x}\right)x+c\left(y+\frac{b}{c}x\right)^2 \\ +(\bar{p}_y-2b\bar{x}-2c\bar{y})\left(y+\frac{b}{c}x\right)+W_0, \end{aligned} \tag{20}$$

where W_0 represents the fixed costs at zero output. In general the smaller the term $(ac-b^2)/c$, the less variable will be the marginal cost of the separate process. This corresponds to the

fact that the more constant the marginal cost of the supplementary process the more closely competitive the products will be as to supply.

Similarly, we can decompose the utility function by considering that there is a common service that can be rendered indifferently by either commodity, but that there is an additional service provided by one (x) but not by the other (y). Thus we may assume for present purposes that either coffee ground at the store or coffee ground on the premises will serve to produce a satisfactory beverage, but that having the coffee ground at the store provides an additional satisfaction in terms of convenience. If these two satisfactions are to be independent of each other, then B/C must be the amount of the x necessary to substitute for one unit of the y in terms of the common satisfaction (note: this is *not* the same as the marginal rate of substitution, as it refers only to a part of the total satisfaction derived from the x). Thus we rewrite (3) as follows:

$$U = -\left(\frac{AC-B^2}{C}\right)x^2 + \left(P_x - \frac{B}{C}P_y + 2\frac{AC-B^2}{C}\bar{x}\right)x - C\left(y + \frac{B}{C}x\right)^2$$

$$+ (P_y + 2B\bar{x} + 2C\bar{y})\left(y + \frac{B}{C}x\right) + U_0, \qquad (21)$$

where U_0 represents the level of utility when no x or y is consumed. Now we see from (18) that this ratio B/C must be less than the ratio b/c obtaining in the common manufacturing process. That is, if in the growing and roasting of coffee the equivalence is pound for pound, then in the brewing of coffee it is necessary, for the Edgeworth phenomenon to occur, that more of the preground coffee be required to produce a given number of cups of beverage of a given quality than of the bean coffee ground on the spot, in this case a not unreasonable hypothesis owing to loss of strength, and possibly wastage where the remainder of a supply becomes unfit for use.

Here then emerges a more or less plausible pattern where the Edgeworth phenomenon would be likely to occur, under competitive conditions. One can envisage a situation in which consumers are on the whole willing to pay a certain premium for the convenience of not having to grind coffee at home, but will

to a large extent forego this convenience if its cost is much increased; under these circumstances a tax on ground coffee, raising the cost of this convenience, might lead to a substantial shift from ground coffee to bean coffee. But since there is less wastage with the bean coffee, the total demand for the growing and roasting of coffee may decline, and is particularly likely to do so if the demand for coffee as a beverage is inelastic. If the supply of coffee is inelastic (as was in fact the case prior to the coffee valorization schemes of Brazil) this diversion to a more efficient (but less satisfying) method of utilization may well reduce the price of the green coffee and with it the price of bean coffee to the point where it would more than offset the tax on that portion that is still pre-ground. More cups of coffee would be consumed, but fewer pounds.

The essential requirement, then, when the matter is analyzed in this way, is that more of the taxed article be required to replace one unit of the untaxed article in providing for the common element of satisfaction to the consumer than can be procured in exchange for one unit of the untaxed article in the common production process. Moreover, the difference between these substitution ratios must be greater, the less perfect is the substitution in consumption between the two commodities, and the more elastic is the output of the common production process. For we can rewrite (14) as follows:

$$Bb > AC - B^2 + Ac$$

$$\frac{b}{c} > \frac{B}{C}\left[1 + \left(\frac{AC - B^2}{B^2}\right)\left(1 + \frac{C}{c}\right)\right] \tag{22}$$

We may take $(AC - B^2)/B^2$ as an index of the degree of imperfect substitution, being zero when the two commodities are perfect substitutes and infinite when they are completely independent; the factor $(1 + C/c)$ reflects the elasticity of demand for the untaxed commodity, relative to that of its supply.

It is not necessary, however, to treat the taxed commodity as the one which undergoes a second process or the one which provides a second type of satisfaction. The untaxed commodity may equally well be regarded in this light. In this case the amount of the taxed commodity equivalent to a unit of the untaxed is

a/b in the common production process, and A/B in the common form of satisfaction; as before, it must require relatively more of the taxed article to take the place of a unit of the untaxed in giving the common satisfaction than is produced in the common process with the resources required to produce one unit of the untaxed item. An example of this type might be plain and creosoted railroad ties: it may require two or three plain ties to give as many years of service as one creosoted tie, and in addition the labor of replacement is saved, whereas the cost of cutting ties may be much the same whether or not the ties are later to be treated with preservative in a separate process. A tax on the commodity that loses in its relative importance as we pass from the common process (cutting) to the common service (rail-bearing) (in this case the plain tie), might well induce the substitution of more treated ties, decrease the number of ties needed for a given amount of service, and if the supply of tie lumber is inelastic, might conceivably drive the price of tie lumber down by more than the tax. This requires of course not only that the supply of tie lumber be inelastic so that a drop in demand will cause a substantial drop in price, but also that the supply of the treatment process be reasonably elastic so that the cost of this processing will not go up too much when the demand is increased as a result of the tax.

In cases of this type we may derive from (18) the following expression for the required difference between the ratios:

$$\frac{a}{b} - \frac{A}{B} > \frac{ac - b^2}{bc} - \frac{AC - B^2}{Bc} \tag{23}$$

Here we see that the required difference becomes larger the less perfect the competition between the commodities either in consumption or in production.

It would also be possible to have a situation where one of the two commodities underwent the extra processing while the other conferred the extra benefit. Realistic examples of this are a bit hard to find, but possibly a number of food preserving practices might come under this heading. The preserving process may be considered to reduce spoilage and thus reduce the amount of raw produce required to provide a given number of calories to

the ultimate consumer, as compared with unpreserved produce. Some of the processes appear to be in varying degrees deleterious, so that fresh produce has a separate utility attached to it in the shape of the absence of these deleterious effects. If consumers were adequately aware of these factors and willing to pay a premium price for produce not so processed, an Edgeworth phenomenon might be obtainable by putting a tax on the non-preserved varieties, inducing a shift to the preserved varieties, reducing the demand for raw produce, and driving the price of raw produce down, possibly by more than the tax.

There is also the symmetrical possibility of a tax raising the price to the buyer by more than the tax, and thus raising the price received by sellers for both the taxed and untaxed items. Instead of (18), for this case we have, for a tax on y, the conditions

$$\begin{aligned}\frac{a}{b} &= \frac{ac-b^2}{bc} + \frac{b}{c} > \frac{b}{c} > \frac{A}{B} + \frac{ac-b^2}{Bc} > \frac{A}{B} \\ &= \frac{AC-B^2}{BC} + \frac{B}{C} > \frac{B}{C}.\end{aligned} \tag{24}$$

The only difference is in the fourth expression, and in the fact that the tax is now on y rather than x.

A possible instance of this sort is the margarine tax: the acreage required to produce a pound of butter appears to be substantially greater than that required to produce a pound of margarine, while the substitution nutritionally is pound for pound; if the demand for table fats is very inelastic, the tax on margarine might mean that more acreage would be required to supply the demand for table fats, and land prices, rural rents, fodder prices, and the price of margarine ingredients might all be increased as a result of the tax. This is to some extent the same configuration that would lead to the possibility of a tax on butter reducing the price of both butter and margarine to the consumer. The one, however, does not necessarily involve the other, though if the commodities are extremely close competitors, it is likely that the two phenomena will occur together or not at all. In this particular instance, however, it appears likely that the demand for table fats is sufficiently elastic and the competition

between margarine and butter for the use of land is sufficiently indirect that the Edgeworth phenomenon is actually not likely to emerge.

This by no means exhausts the possibilities, but it does provide a few hints as to where to look for the phenomenon and what its nature is. In particular, although mathematically it is possible to substitute for interdependent commodities properly selected "baskets" that are mutually independent (at least in a small neighborhood), this mathematical process need not correspond to any identifiable "satisfactions" or "processes" as we have here assumed, and it would be quite possible for the Edgeworth phenomenon to occur without such mathematical breakdowns having any obvious real counterpart. Moreover, we have examined only the case of two interrelated commodities, and it is clear that if we allow for more complex interrelations among several commodities additional possibilities may occur. But these simple problems may serve to provide some slight further insight into the problems of tax incidence.

None of the foregoing serves to invalidate in any way the proposition that from an inclusive viewpoint excise taxes that produce divergences between marginal cost and price to the consumer impair the general welfare. All that has been shown is that under some circumstances the imposition of a tax may redound to the net benefit of some of the parties to the taxed transaction, in this case, the consumers. The other parties (here, the sellers) will in general lose by more than the sum of the benefit to the gainers and the revenue to the government. Thus unless the sellers are deemed in some way less worthy of consideration than the buyers, such a tax is still not a desirable one from the point of view of the community as a whole, nor can the discovery of an instance of Edgeworth's phenomenon be considered necessarily "welcome news to the harassed minister of finance." For we can write for the excess of utilities over costs:

$$(25)\quad \begin{aligned} S &= U - W \\ &= \bar{S} - (A+a)(x-\bar{x})^2 - (B+b)(x-\bar{x})(y-\bar{y}) - (C+c)(y-\bar{y})^2 \end{aligned}$$

This function is thus at a maximum when $x = \bar{x}$ and $y = \bar{y}$.

That this is a true maximum and not a minimax is assured by the fact that D is positive.

Even if we were to admit that the buyers for some reason should be benefitted at the expense of the sellers, it would in general be desirable to levy a direct tax on the rents, income, or other surplus of the sellers and pay a direct subsidy to the buyers, rather than achieve the redistribution by means of an excise tax. To be sure, if some sort of excise is to be used as the means of achieving this redistribution, then a tax in an Edgeworthian situation would be relatively more effective in achieving this redistribution than a tax on an independent commodity where the amount of redistribution would necessarily be less than the tax revenue. But in general an excise tax cannot be justified by such an argument except where the person who is to bear the burden would otherwise be beyond the reach of the taxing authority. The usual case of this sort is when the seller is a foreigner. In the coffee case cited above, the redistribution carried out is one in favor of the domestic coffee drinker at the expense of the Brazilian coffee grower. But this is in reality no more than a peculiarly effective instance of monopsonistic exploitation by the taxing country, and unless such action is taken as a means of adjusting disequilibria in international trade, it could well be considered a form of economic warfare.

In short it is wrong to dismiss the Edgeworth paradox as something that can happen only in rare and peculiar circumstances. The phenomenon is not frequent, to be sure, but there seems little doubt but that cases can occur and that the circumstances tending to produce the phenomenon can be recognized if one knows what to look for. On the other hand, even when found, instances of the phenomenon are far from the "grateful boon" that Seligman thought that they would provide, did they exist. At least in domestic matters, existence of an Edgeworth phenomenon in no way justifies a departure from normal standards of tax policy. It is only where the analysis relates to the foreign policy of a nationalistic state, bent on deriving the maximum advantage from its strategic position, that the discovery and exploitation of the phenomenon becomes of concern.

CHARLES E. FERGUSON

Modified Edgeworth Phenomena and the Nature of Related Commodities

I. *Introduction*

In one of his most celebrated contributions to economic theory,[1]) Professor Hotelling analyzed the so-called Edgeworth Taxation Paradox, *viz.*, the proposition that an excise tax imposed upon one of two related commodities may result in a decline in the price of both. Far from being a mere curiosum as Edgeworth suggested, he found that the conditions of the Paradox are satisfied in a wide class of cases conforming to certain theoretical requirements which he adduced for demand and supply functions. More specifically, Professor Hotelling proved that in both competitive[2]) and monopolistic markets, the Edgeworth phenomenon is possible if the two commodities are substitutes in both consumption and production.

At this quarter-century anniversary of Professor Hotelling's important paper, it seems appropriate to extend his analysis to modified Edgeworth cases. In particular, we investigate the consumption and production relationships that must prevail if an excise tax imposed upon one commodity results (a) in a decrease in its price and an increase in the price of the other (related) commodity, and (b) in an increase in the price of the taxed commodity but a decrease in the price of the related product. As a convention, and without loss of generality, throughout the paper we assume that the tax in question is imposed on good X_1.

[1]) Harold Hotelling, "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions," *Journal of Political Economy*, XL (1932), pp. 577—616.

[2]) "Competition" is used here in the Cournot sense of competition between two monopolists.

II. *Analytical Framework*

Consider two related commodities, X_1 and X_2, whose demand and supply functions are given by

$$(1) \qquad p_i = f_i(x_1, x_2) \text{ and } p_i = g_i(x_1, x_2) \qquad (i = 1, 2)$$

respectively; or, assuming that the Jacobian of transformation does not vanish, by the inverse relations

$$x_i = F_i(p_1, p_2) \text{ and } x_i = G_i(p_1, p_2) \qquad (i = 1, 2).$$

We assume that these functions satisfy the theoretical requirements prescribed by Professor Hotelling for the case of unlimited budgets. Denoting partial derivation by an additional subscript, these requirements are, in terms of equations (1):[3])

$$(2) \qquad f_{ij} = f_{ji},\ g_{ij} = g_{ji},\ [f_{ij}] \ll 0,\ [g_{ij}] \gg 0.$$

Furthermore, we adopt the definition of complementarity suggested by Professor Hotelling and others.[4]) Specifically, X_1 and X_2 are competitive or complementary according as $F_{12} = F_{21} \gtrless 0$. However, since we will use equations in the form displayed in (1), it is necessary to express this definition of complementarity in terms of the f- and g-functions. Let

$$J = \left| \frac{\partial(x_1, x_2)}{\partial(p_1, p_2)} \right|.$$

Then obtaining the inverse of the matrix of partial derivatives by the adjoint method, we can display the relationship between the partial derivatives of the quantities with respect to the prices and the partials of the prices with respect to the quantities:

$$F_{11} = Jf_{22},\ F_{12} = F_{21} = -Jf_{21} = -Jf_{12},\ F_{22} = Jf_{11}.$$

Since we have assumed that the quadratic form is negative definite,[5]) $J > 0$. Hence the sign of $f_{12} = f_{21}$, is the opposite of the sign of $F_{12} = F_{21}$.

[3]) Hotelling, *op. cit.*, p. 591 and p. 597. More specifically, the two important properties are: (a) the "symmetry" or "integrability" conditions; and (b) negative (positive) definiteness of the quadratic form of the coefficients of the demand (supply) functions.

[4]) *Ibid.*, p. 599. See also Oscar Lange, "Complementarity and Interrelations of Shifts in Demand," *Review of Economic Studies*, VIII (1940—41), 59.

[5]) This assumption is expressed in relations (2) for the f- and g-functions.

Accordingly, X_1 and X_2 are competitive or complementary in consumption according as $f_{12} \lessgtr 0$. Similarly, by an extension of the Hotelling definition, the goods are competitive or complementary in production according as $G_{12} = G_{21} \lessgtr 0$. Consequently, X_1 and X_2 are also competitive or complementary in production according as $g_{12} = g_{21} \gtrless 0$.

Before proceeding to the analysis of cases, it is convenient to establish the equations expressing the change in commodity price that is attributable to an excise tax. This is done for both competitive and monopolistic markets.

Let $h_i = f_i - g_i$, so that

$$h_{ij} = h_{ji} = \partial h_i/\partial x_j = \partial h_j/\partial x_i,$$

and designate the accompanying Jacobian by

$$D = \left| \frac{\partial(h_1, h_2)}{\partial(x_1, x_2)} \right| > 0,$$

since the elements of D are the differences between the corresponding elements of a negative definite and a positive definite quadratic form. If p_i and x_i are the equilibrium values of price and quantity that bring competitive demand and supply into equality, then

$$h_i(x_1, x_2) = 0 \qquad (i = 1, 2). \tag{3}$$

Assume that a small excise tax of t_1 per unit is imposed upon X_1 and designate the resulting prices and quantities by $p_i + \Delta p_i$, $x_i = \Delta x_i$. In this situation, the demand price of the taxed commodity must exceed its supply price by the amount of the tax. That is,

$$h_1(x_1+\Delta x_1, x_2+\Delta x_2) = t_1, \quad h_2(x_1+\Delta x_1, x_2+\Delta x_2) = 0. \tag{4}$$

Subtracting equation (3) from equation (4) and applying the definition of "derivative", we obtain

$$\sum_{j=1}^{2} h_{ij} \Delta x_j = \delta_{1i} t_1, \qquad (i = 1, 2) \tag{5}$$

where δ_{1i} is the Kronecker delta. Equations (5) are simultaneous linear equations which are solvable by Cramer's Rule for Δx_i:

$$\Delta x_i = \frac{1}{D} \begin{vmatrix} t_1 & h_{12} \\ 0 & h_{22} \end{vmatrix}, \quad \Delta x_2 = \frac{1}{D} \begin{vmatrix} h_{11} & t_1 \\ h_{21} & 0 \end{vmatrix}. \tag{6}$$

From equation (1),

$$\Delta p_i = \sum_{j=1}^{2} f_{ij} \Delta x_j. \qquad (i = 1, 2)$$

Substituting the expressions for Δx_i from equation (6),

$$\Delta p_1 = \frac{t_1}{D} (f_{11} h_{22} - f_{12} h_{21}),$$

$$\Delta p_2 = \frac{t_1}{D} (f_{21} h_{22} - f_{22} h_{21}).$$

Finally, substituting the expression $f_{ij} - g_{ij}$ for h_{ij} and recalling that $t_1 > 0$ and $D > 0$, we can write the following expression relating the direction of change in price to the imposition of an excise tax on the first commodity:

(7a) $\quad \text{sign } (\Delta p_1) = \text{sign } (f_{11} f_{22} - f_{11} g_{22} - f_{12}^2 + f_{12} g_{21}),$

(7b) $\quad \text{sign } (\Delta p_2) = \text{sign } (f_{22} g_{21} - f_{21} g_{22}).$

Equations (7) express the price changes attributable to an excise on X_1 when the two related commodities are produced by competing entrepreneurs. We turn next to the situation that prevails when both commodities are produced by a single monopolist. As Professor Hotelling has pointed out, in the monopoly situation the supply relationship between commodities is immaterial. Accordingly, we investigate the consumption relationships under monopoly using the convenient fiction of costless production.

After the imposition of an excise tax on the first commodity, the monopolist maximizes the resulting profit equation

$$p_1 x_1 + p_2 x_2 - t_1 x_1 = \pi - t_1 x_1,$$

thus determining quantities so that

$$\pi_1 = t_1, \ \pi_2 = 0, \text{ and } [\pi_{ij}] \ll 0.$$

By differentiating the two first-order conditions with respect to the tax rate, we obtain

$$\sum_{j=1}^{2} \pi_{ij} x_{jt} = \delta_{1i}, \qquad (i = 1, 2)$$

where δ_{1i} is a Kronecker delta. These are two linear equations

which may be solved for the unknown tax derivatives. Applying Cramer's Rule,

$$x_{jt} = \frac{\text{cofactor of } \pi_{1j}}{|[\pi_{ij}]|} = \varphi_{ij}. \qquad (j = 1,\ 2)$$

Then using equation (1), the price changes corresponding to a tax on X_1 can be determined:

$$p_{it} = \sum_{j=1}^{2} f_{ij} x_{jt} = \sum_{j=1}^{2} f_{ij} \varphi_{1j}, \qquad (i = 1,\ 2)$$

or alternatively:

(8a) $$\operatorname{sign}(p_{1t}) = \operatorname{sign}(f_{11}\varphi_{11} + f_{12}\varphi_{12}),$$

(8b) $$\operatorname{sign}(p_{2t}) = \operatorname{sign}(f_{21}\varphi_{11} + f_{22}\varphi_{12}).$$

Equations (1), (2), (7), and (8) and the definition of complementarity provide all that we need of a purely formal nature.

III. *Modified Edgeworth Phenomena in a Competitive Situation*

Professor Hotelling has shown that a necessary condition for the Edgeworth Paradox in a competitive market is that the two commodities compete in both consumption and production.[6]) We now investigate the possible commodity relationships for modified cases of the Edgeworth Paradox. In doing so, we utilize the convenient feature that the possible existence, in any specific case, of the modified Edgeworth phenomena can be established by providing an example of demand and supply functions which meet all requirements of the situation. On the other hand, to establish the impossibility of the phenomena requires a formal proof.

A. Case I: Price of X_1 Lowered, Price of X_2 Raised.

The first modified case considered is that in which an excise tax imposed upon X_1 results in a decrease in its price, accompanied by an increase in the price of the related commodity X_2. From equations (7), this requires that

(9a) $$f_{11}f_{22} - f_{12}^2 < f_{11}g_{22} - f_{12}g_{21},$$

(9b) $$f_{22}g_{12} > f_{12}g_{22}.$$

[6]) Hotelling, *op. cit.*, p. 604.

Since, from inequalities (2), $f_{11}f_{22} - f_{12}^2 > 0$, $f_{11} < 0$, and $g_{22} > 0$, to satisfy (9a) we must have $-f_{12}g_{21} > 0$. This requires that either $f_{12} < 0$ or $g_{21} < 0$, but not both.

Consequently, if $g_{21} < 0$, then $f_{12} > 0$. Using this information in (9b), together with the knowledge that $f_{22} < 0$ and $f_{22}g_{12} > 0$, we see that $f_{12}g_{22} > 0$. Therefore, it is possible for Case I to exist if the two commodities are complementary in both consumption and production. Indeed, an example of this which satisfies all requirements is given by the following set of equations:[7])

$$\text{Demand}\begin{cases} p_1 = 11 - x_1 + 2x_2, \\ p_2 = 30 + 2x_1 - 5x_2, \end{cases} \qquad \text{Supply}\begin{cases} p_1 = 1 + 5x_1 - 2x_2, \\ p_2 = 10 - 2x_1 + x_2. \end{cases}$$

On the other hand, since the signs of f_{12} and g_{21} cannot be the same, it is impossible to obtain the desired results if the two commodities are competitive in consumption and complementary in production, or *vice versa*. Furthermore, Case I is impossible if the commodities are substitutes in both consumption and production. This situation requires $f_{12} < 0$ and $g_{12} > 0$. That is, it is necessary to satisfy inequalities (9) subject to the additional inequalities:

$$f_{12} < 0, g_{12} > 0, f_{11} < 0, f_{22} < 0, g_{11} > 0, g_{22} > 0, f_{11}f_{22} - f_{12}^2 > 0.$$

The last inequality above implies

$$\text{(10)} \qquad \frac{f_{11}f_{22}}{f_{12}^2} > 1.$$

Now since $f_{22} < 0$ and $f_{12} < 0$ by the special hypothesis, the second inequality in (9b) may be written $|f_{22}g_{12}| < |f_{12}g_{22}|$. This, in turn, implies

[7]) The procedure for obtaining examples of Edgeworth or modified Edgeworth phenomena is as follows: determine the direction of the inequalities relevant for equations (7); then select f_{ij} and g_{ij} values to satisfy these inequalities. The $f_{ij}(g_{ij})$ values are the coefficients of x_i in the demand (supply) equations. Appropriate constants may be selected by solving the demand and supply equations simultaneously. To check the equations, introduce an excise tax on one of the commodities. Remembering that for the taxed commodity, demand price must exceed supply price by the amount of the tax, the demand and supply equations, properly adjusted, may be solved simultaneously for p_i and x_i, both expressed in terms of the tax rate. Differentiating the expressions for p_i with respect to the tax rate then confirms the result.

$$(11) \qquad f_{22}/f_{12} < g_{22}/g_{12}.$$

In light of (10), the inequality in (9a) may be written $|f_{11}g_{22}| < |f_{12}g_{21}|$, or

$$(12) \qquad f_{11}/f_{12} < g_{21}/g_{22}.$$

Multiplying (11) by (12), we have

$$\frac{f_{11}f_{22}}{f_{12}^2} < \frac{g_{22}}{g_{12}}\frac{g_{12}}{g_{22}} = 1,$$

thus contradicting the special hypothesis embodied in inequality (10).

B. Case II: Price of X_1 Raised, Price of X_2 Lowered.

The second modification that we investigate is the case in which an excise tax levied on the first commodity results in an increase in its price but a decrease in the price of the related good. From equations (7), this result is possible if

$$(13a) \qquad f_{11}f_{22} - f_{12}^2 > f_{11}g_{22} - f_{12}g_{21},$$

$$(13b) \qquad f_{22}g_{12} < f_{12}g_{22}.$$

Inequalities (13) are satisfied by three different sets of commodity relationships. In the first place, if the two goods are substitutes both in consumption and production, Case II is satisfied by the following set of equations:

$$\text{Demand}\begin{cases} p_1 = 41 - 5x_1 - 2x_2, \\ p_2 = 21 - 2x_1 - 2x_2, \end{cases} \quad \text{Supply}\begin{cases} p_1 = 1 + 9x_1 + 4x_2, \\ p_2 = 1 + 4x_1 + 2x_2. \end{cases}$$

Secondly, Case II is also possible when the commodities are complementary in consumption and competitive in production. An example of demand and supply functions satisfying these conditions is given by

$$\text{Demand}\begin{cases} p_1 = 12 - 2x_1 + x_2, \\ p_2 = 11 + x_1 - x_2. \end{cases} \quad \text{Supply}\begin{cases} p_1 = 3 + x_1 + x_2, \\ p_2 = 2 + x_1 + 2x_2. \end{cases}$$

Finally, if the two commodities are complements in both consumption and production, the modified Edgeworth results can be obtained. An example meeting these conditions is

$$\text{Demand}\begin{cases} p_1 = 20 - 5x_1 + 4x_2, \\ p_2 = 18 + 4x_1 - 4x_2, \end{cases} \quad \text{Supply}\begin{cases} p_1 = 6 + 3x_1 - 3x_2, \\ p_2 = 4 - 3x_1 + 4x_2. \end{cases}$$

On the other hand, Case II is not possible if the commodities are substitutes in consumption and complements in production. For if this were so, $f_{12} < 0$ and $g_{12} < 0$, implying $f_{22}g_{12} > 0$ and $f_{12}g_{22} < 0$. But these two inequalities are inconsistent with (13b), and, accordingly, the special hypothesis must be rejected.

IV. *Modified Edgeworth Phenomena in a Monopoly Situation*

We now investigate the feasibility of the modified Edgeworth phenomena in a monopoly situation, using the same two cases discussed in Section III above. As stated previously, Professor Hotelling has proved that a tax on one commodity can result in a decrease in the price of both if the goods are substitutes in consumption, the production relationship being less material under monopolistic organization.

A. Case I.

From equations (8), an excise tax imposed upon X_1 will result in a decrease in its price and an increase in the price of X_2 if[8])

(14a) $$f_{11}\varphi_{11} + f_{12}\varphi_{12} < 0,$$

(14b) $$f_{12}\varphi_{11} + f_{22}\varphi_{12} > 0.$$

Of the two possible relationships, Case I may exist if the commodities are complementary in consumption. This is illustrated by the following quadratic demand functions:[9])

[8]) It should be recalled that we have the additional information that

$$f_{ii} < 0,\ \varphi_{ii} < 0,\ |[f_{ij}]| > 0, \text{ and } |[\varphi_{ij}]| > 0.$$

[9]) The procedure involved in obtaining examples for a monopolistic market can be sketched briefly. Determine the relevant inequalities for equations (8) and select a set of values for f_{ij} and φ_{ij} which satisfy these inequalities and the theoretical requisites of demand functions. Next obtain the three different equations for π_{ij}; these equations involve x_i, f_{ij} and f_{ijk}. Using the previously assumed values of f_{ij} and postulating equilibrium values for x_i, these equations may be solved recursively for f_{ijk}. Then postulate two quadratic demand functions with unknown coefficients and constants. Differentiating these quadratic functions twice, we obtain expressions for f_{ijk} in terms of the coefficients of the demand functions. Consequently, since values have previously been determined for f_{ijk}, one may obtain numerical solutions for the coefficients. Next substitute the numerical values into the demand functions, and the latter into the profit equation of the monopolist. Then state the first-order conditions for maximum profit and substitute the assumed equilibrium values of x_i into these equations. The resulting expressions may be solved for the constants of the demand equations. The functions thus obtained may be checked by

$$p_1 = \tfrac{1}{2}x_1^2 - x_1x_2 - 5x_1 + 3x_2 + \tfrac{11}{2},$$
$$p_2 = -\tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 + 3x_1 - 2x_2 - 2.$$

On the other hand, Case I is impossible if the commodities are competitive in consumption. For if so, $f_{12} < 0$. Then with $f_{ii} < 0$ and $\varphi_{ii} < 0$, in order to satisfy (14a), we must have $\varphi_{12} > 0$. But then $|f_{11}\varphi_{11}| < |f_{12}\varphi_{12}|$, or

$$f_{11}/f_{12} < -\varphi_{12}/\varphi_{11}. \tag{15}$$

To satisfy (14b) it is necessary that $|f_{12}\varphi_{11}| > |f_{22}\varphi_{12}|$, or

$$f_{22}/f_{12} < -\varphi_{11}/\varphi_{12}. \tag{16}$$

Multiplying (15) by (16), we obtain

$$\frac{f_{11}f_{22}}{f_{12}^2} < \frac{\varphi_{12}\varphi_{11}}{\varphi_{11}\varphi_{12}} = 1. \tag{17}$$

But since f_{ij} are the coefficients of a negative definite quadratic form,

$$f_{11}f_{22}/f_{12}^2 > 1. \tag{18}$$

Since (17) and (18) are inconsistent, the hypothesis is contradicted.

B. Case II.

In some ways, Case II in the monopoly situation is the simplest of all cases, for, irrespective of the commodity relationship that prevails, this modified Edgeworth phenomenon is possible. That is, Case II may occur if the commodities are either complementary or competitive in consumption. Examples of the two situations, in the order designated, are provided by the following sets of demand functions:

$$\begin{cases} p_1 = \tfrac{1}{2}x_1^2 - x_1x_2 - x_2^2 - x_1 + 4x_2 - \tfrac{3}{2}, \\ p_2 = -\tfrac{1}{2}x_1^2 - 2x_1x_2 + \tfrac{5}{2}x_2^2 + 4x_1 - 5x_2 + 2; \end{cases}$$

$$\begin{cases} p_1 = -\tfrac{1}{2}x_1^2 - x_1x_2 - \tfrac{1}{2}x_2^2 + x_1 + x_2 + 2, \\ p_2 = -\tfrac{1}{2}x_1^2 - x_1x_2 - \tfrac{1}{2}x_2^2 + x_1 + 4. \end{cases}$$

postulating a tax, and then differentiating the first-order conditions with respect to the tax rate. Next assume that the tax is an *incipient* excise and substitute the assumed equilibrium values of x_i into the π_{ii} equations, obtaining two simultaneous equations in the unknowns x_{it}. These may be solved and substituted into the expressions for f_{ij}, completing the check.

V. *Summary*

We are now in a position to summarize the results of this investigation.

A. Competition:

1. Professor Hotelling has proved that the two related commodities must be competitive in both consumption and production if the prices of both are to be lowered by the imposition of an excise tax on one of the goods.

2. Price of taxed commodity lowered, price of other good raised:

a. this situation can result if the two commodities are complementary in both consumption and production;

b. impossible if commodities are substitutes in both consumption and production, or if they are *either* complements in production and substitutes in consumption *or* substitutes in production and complements in consumption.

3. Price of taxed good increased, price of other decreased:

a. possible if the two commodities are substitutes in both consumption and production, complements in consumption and substitutes in production, or complements in both consumption and production;

b. impossible if the two commodities are substitutes in consumption and complements in production.

B. Monopoly:

1. Professor Hotelling has proved that the price of both commodities may be decreased by a tax on one of them provided the commodities are substitutes in consumption.

2. Price of taxed commodity lowered, price of other raised:

a. possible if commodities are complementary in consumption;

b. impossible if commodities are substitutes in consumption.

3. Price of taxed commodity raised, price of other lowered: this case is possible irrespective of the consumption relationship which prevails between the commodities.

PART V

On Econometric Methods

RAGNAR FRISCH

Market Prices vs. Factor Costs and the Constancy of Production Coefficients

The question of whether it is most appropriate to use market prices or factor costs in national income statistics and more generally in economic model work, has been much discussed. I shall offer some comments on this question. Or rather, I shall look at this whole question from a somewhat more general viewpoint more adaptable for decision model work. Instead of factor input as distinguished from indirect taxes (minus subsidies), I shall consider the *proportional* input elements as distinguished from the residuum.

I will try to bring out that in so far as the *global national product* is concerned, the effects of including only the (proportional) factor costs instead of including the total market values, *disappear* to a large extent when the values are *deflated* so as to bring out the volume figures instead of the value figures.

In so far as the *production coefficients* are concerned, the difference between alternative kinds of figures is more complicated. We have to distinguish between at least four different types of magnitudes: *strictly physical* quantities, *volume* figures, *semi volume* figures and *current values.* For each of these types we must look into the question of the constancy of the coefficients.

For practical reasons the usual input output coefficients are as a rule computed as ratios between market values observed in a base year. This is also done in Norwegian work and in a general way I agree to this procedure for the reasons given in the sequel. There are, however, special purposes for which some modifications may be used. Compare the comments below on the method followed in the Oslo median model.

A. *The Strictly Physical Structure*

To bring out the essence of the problem as it appears in a

decision model, let us first consider a table with all final deliveries aggregated, and with only strictly physical quantities involved so that no vertical summations are possible. The result of such a set up is given in Table 1. The strictly physical quantities are denoted by small letters.

TABLE 1

Input-Output of Strictly Physical Quantities

		Receiving sector No.		Final delivery	Total delivery
		$h = 1$	2		
Delivering sector No.	$k = 1$	0	x_{12}	x_{1*}	x_1
	2	x_{21}	0	x_{2*}	x_2
Primary input	Labour	w_1	w_2	—	—
	Non competitive imports	b_1	b_2	—	—

If we do not impose any other relations than the definitions of the total products, i.e.,

$$
\begin{aligned}
x_{12} + x_{1*} &= x_1 \\
x_{21} + x_{2*} &= x_2
\end{aligned} \tag{1}
$$

we have $10 - 2 = 8$ degrees of freedom.

If we introduce the 6 production coefficients by

$$
\begin{aligned}
x_{12} &= x'_{12} x_2 & x_{21} &= x'_{21} x_1 \\
w_1 &= w'_1 x_1 & w_2 &= w'_2 x_2 \\
b_1 &= b'_1 x_1 & b_2 &= b'_2 x_2
\end{aligned} \tag{2}
$$

and for the moment consider all these coefficients as variables, we have 16 variables and 8 equations, hence still 8 degrees of freedom. As basis variables we can choose for instance the 6 coefficients and x_1, x_2. Or we can choose the 6 coefficients and x_{1*}, x_{2*}. The 8 basis equations are in the first case (2) together with

$$
x_{1x} = x_1 - x'_{12} x_2 \qquad x_{2x} = -x'_{21} x_1 + x_2. \tag{3}
$$

In the second case they are

$$
x_1 = \frac{x_{1*} + x'_{12} x_{2*}}{1 - x'_{12} x'_{21}} \qquad x_2 = \frac{x'_{21} x_{1*} + x_{2*}}{1 - x'_{12} x'_{21}} \tag{4}
$$

together with the 6 expressions obtained by inserting (4) into (2).

This is the structure of the system expressed in strictly physical terms. The case of constant coefficients is covered simply by putting the coefficients in the basis equations equal to their constant values. This leaves us with 2 degrees of freedom, which may, for instance, be unfolded by x_1, x_2 or by x_{1*}, x_{2*}.

B. *The Structure Expressed in Volume Figures*

Now let us introduce a set of *product prices* π_1, π_2 and *factor prices* π_w, π_b. We call them *standard* prices. Let us see how the various constellations of the system which are physically possible with the degrees of freedom in (1), can be expressed in the value terms derived from the standard prices. We also introduce residual items ε_1, ε_2 in the two production sectors. The residual items may be the sum of taxes T_h and net profits (savings) δ_h. For practical purposes in a decision model these residual items are very important, but their introduction causes considerable complications in the definitional set up. These difficulties we must consider in a systematic way.

The new figures are listed in Table 2. We could, if we wanted to, introduce different wage rates in the two sectors and also different import prices, but that is unessential in the present connection.

TABLE 2

Interflow Table of Values Reckoned at Standard Prices and with Standard Residuum Elements

		Receiving sector No. $h=1$	2	Final delivery	Total delivery
Delivering sector No.	$h=1$	0	$\pi_1 x_{12}$	$\pi_1 x_{1*}$	$\pi_1 x_1$
	2	$\pi_2 x_{21}$	0	$\pi_2 x_{2*}$	$\pi_2 x_2$
Primary input	Labour	$\pi_w w_1$	$\pi_w w_2$	$-\pi_w(w_1+w_2)$	0
	Non competitive imports	$\pi_b b_1$	$\pi_b b_2$	$-\pi_b(b_1+b_2)$	0
Residual input		ε_1	ε_2	$-(\varepsilon_1+\varepsilon_2)$	0
Grand total		$\pi_1 x_1$	$\pi_2 x_2$	0	$\pi_1 x_1+\pi_2 x_2$

The prices π_1 and π_2 are actual prices per physical unit, say per kilogram or per kWh. The wage rate π_w is also reckoned per physical unit, say per hour of work. Similarly for π_b. The residual input is measured in money.

It should be understood that Tables 1 and 2 exist simultaneously, and that the actual physical quantities in the two tables are the same.

In Table 2 we have imposed the condition that the sum in column No. h shall be equal to $\pi_h x_h$. This is equivalent with defining the residual ε_h when the prices are given. Or we may inversely consider the condition as defining the prices π_h in terms of the residual inputs ε_1, ε_2. We will most of the time adopt the latter viewpoint.

The column sum conditions are expressed by

$$\begin{aligned} \pi_2 x_{21} + \pi_w w_1 + \pi_b b_1 + \varepsilon_1 &= \pi_1 x_1 \\ \pi_1 x_{12} + \pi_w w_2 + \pi_b b_2 + \varepsilon_2 &= \pi_2 x_2. \end{aligned} \tag{5}$$

We define the residual *rates* $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ by

$$\varepsilon_k = \bar{\varepsilon}_k x_k \quad (k = 1,\ 2). \tag{6}$$

If need be, these residual rates will be called *direct* residual rates to distinguish them from certain *aggregate* residual rates to be considered later.

This gives 4 equations in addition to the 8 we had before. The additional variables are

(7)

Number of variables	
4	$\pi_1,\ \pi_2,\ \pi_w,\ \pi_b$
2	$\varepsilon_1,\ \varepsilon_2$
2	$\bar{\varepsilon}_1,\ \bar{\varepsilon}_2$
8	Total

In other words we have $8 - 4 = 4$ more degrees of freedom than in the table of strictly physical quantities. As additional basis variables we choose the factor prices π_w, π_b and the residual rates $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$. Using x_1, x_2 rather than x_{1*} and x_{2*} as basis variables, the total set of basis variables will be

$$
\begin{array}{ll}
 & x_1,\ x_2 \\
 & \pi_w,\ \pi_b \\
(8) & \bar{\varepsilon}_1,\ \bar{\varepsilon}_2 \\
 & x'_{12},\ x'_{21},\ w'_1,\ w'_2,\ b'_1,\ b'_2.
\end{array}
$$

If all the 6 coefficients listed on the last row in (8) are taken as given, there are 6 basis variable left, namely those on the first three rows of (8). Of these x_1, x_2 determine the physical constellation of the system — all the other physical features following from x_1, x_2 — while π_w, π_b, $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$ determine the price features — the other prices following from π_w, π_b, $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$.

The prices π_1 and π_2 as functions of the coefficients and the basis price elements, are determined by inserting into (5) from (2) and (6) which gives the system of two equations

$$
(9) \qquad \begin{aligned}
\pi_1 - \pi_2 x'_{21} &= \pi_w w'_1 + \pi_b b'_1 + \bar{\varepsilon}_1 \\
-\pi_1 x'_{12} + \pi_2 &= \pi_w w'_2 + \pi_b b'_2 + \bar{\varepsilon}_2.
\end{aligned}
$$

The solution of this is

$$
(10) \qquad \begin{aligned}
\pi_1 &= \frac{\pi_w(w'_1 + w'_2 x'_{21}) + \pi_b(b'_1 + b'_2 x'_{21}) + (\bar{\varepsilon}_1 + \bar{\varepsilon}_2 x'_{21})}{1 - x'_{12} x'_{21}} \\
\pi_2 &= \frac{\pi_w(w'_1 x'_{12} + w'_2) + \pi_b(b'_1 x'_{12} + b'_2) + (\bar{\varepsilon}_1 x'_{12} + \bar{\varepsilon}_2)}{1 - x'_{12} x'_{21}}
\end{aligned}
$$

In the case of n sectors (9) has the form

$$
(11) \qquad \sum_{k=1}^{n} \pi_k (\delta - x')_{kh} = \pi_w w'_h + \pi_b b'_h + \bar{\varepsilon}_h \qquad (h = 1,\ 2 \ldots n)
$$

which is solved by

$$
(12) \quad \pi_k = \sum_{h=1}^{n} (\pi_w w'_h + \pi_b b'_h + \bar{\varepsilon}_h)(\delta - x')^{-1}_{hk} \qquad (k = 1,\ 2, \ldots n).
$$

The formulae (10) show that if not only the factor prices, but also the direct residual rates are constant, the product prices will also be constant. In other words the whole structure of standard prices will be fixed. We consider them as *base prices* and take the corresponding values as defining *volume figures.*

We put

(13) $X_k = \pi_k x_k \qquad X_{k*} = \pi_k x_{k*}$

(14) $X_{kh} = \pi_k x_{kh}$

(15) $W_h = \pi_w w_h \qquad B_h = \pi_b b_h.$

The volume figures defined by (13)—(15) are entered in Table 3. That is to say, if the numerical figures are entered, Table 2 and Table 3 will be exactly the same.

Keeping the price structure — as defined through π_w, π_b, $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$ — *constant* and varying x_1 and x_2, we get different constellations of the volume figures. Since it is simply a question of units of measurement to pass from the system of strictly physical quantities to the volume figures, we can just as well think of X_1 and X_2 as varying under constant π_w, π_b, $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$.

TABLE 3

Interflow Table of Volume Figures Reckoned under Base Year Prices and Base Year Residual Elements

		Receiving sector No. $h = 1$	2	Final deliveries	Total deliveries
Delivering sector No.	$k = 1$	0	X_{12}	X_{1*}	X_1
	2	X_{21}	0	X_{2*}	X_2
Primary input	Labour	W_1	W_2	$-(W_1+W_2)$	0
	Non competitive imports	B_1	B_2	$-(B_1+B_2)$	0
Residual input		ε_1	ε_2	$-(\varepsilon_1+\varepsilon_2)$	0
Grand total		X_1	X_2	0	X_1+X_2

The introduction of the volume figures, i.e. the magnitudes denoted by capital letters in (13)—(15), does not change the number of degrees of freedom because to each new magnitude corresponds one definitional equation. If we assume that in (8) not only x_1 and x_2, but also $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ are changing while the factor prices π_w and π_b as well as the 6 physical quantity coefficients are given, we have 4 degrees of freedom. (In n sectors $2n$ degrees of freedom). From the discussion in the sequel it will appear that we will reserve the terminology volume figures, as

defined through (13)—(15), to the case of *constant* residual rates. These constant rates we can assume as *observed* by the actual situation in a base year. Now there remain only two degrees of freedom in the volume figures. They may be unfolded say by X_1 and X_2. (In n sectors n degrees of freedom).

Table 3 has at the same time the following two properties: (1) the magnitudes entering into it have the character of *volume* figures (because they represent values at base year prices), and (2) vertical summations in the table are possible.

I shall look a little closer into the particular aspect of the question that is represented by the constancy of the residual rates.

C. *Constant Factor Prices and Constant Residual Rates Entail Constant Input-Output Volume Coefficients.*

We define

$$X'_{kh} = \frac{X_{kh}}{X_h} \tag{16}$$

$$W'_h = \frac{W_h}{X_h} \qquad B'_h = \frac{B_h}{X_h}. \tag{17}$$

From the definitions (13)—(15) follows that the coefficients (16)—(17) are equal to

$$X'_{kh} = \frac{\pi_k}{\pi_h} x'_{kh} \tag{18}$$

$$W'_h = \frac{\pi_w}{\pi_h} w'_h \quad B'_h = \frac{\pi_b}{\pi_h} b'_h \tag{19}$$

where the π_k are given by (12), and π_w, π_b are the given factor prices. If all the production coefficients x'_{12}, x'_{21}, w'_1, w'_2 etc. reckoned in strictly physical quantities are constant, we see from (10) and (16)—(17) that constant factor prices and constant residual rates $\bar{\varepsilon}_k$ entail constant X'_{kh}, W'_h and B'_h.

The conditions of constant residual rates $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ can be transformed into corresponding conditions about the residual rates expressed as fractions of X_1 and X_2. Indeed, from (6) and (13), we get

$$\varepsilon_k = \varepsilon'_k X_k \tag{20}$$

where

$$\varepsilon'_k = \frac{\bar{\varepsilon}_k}{\pi_k}. \tag{21}$$

The last formula taken in conjunction with (10) shows immediately that if we have constant factor prices and constant quantity coefficients, constant $\bar{\varepsilon}_k$ rates will entail constant ε'_k rates.

On the other hand, if the marginal rates ε'_k are given, instead of the $\bar{\varepsilon}_k$, we deduce from (9) by inserting from (21)

$$\begin{aligned} \pi_1(1 - \varepsilon'_1) - \pi_2 x'_{21} &= \pi_w w'_1 + \pi_b b'_1 \\ -\pi_1 x'_{12} + \pi_2(1 - \varepsilon'_2) &= \pi_w w'_2 + \pi_b b'_2. \end{aligned} \tag{22}$$

From the equations (22) the π_k are determined. That is to say: Constant coefficients in the strictly physical structure, constant factor prices and constant residual rates ε'_k entail constant prices π_k and hence by (21) constant $\bar{\varepsilon}_k$.

It is the same set of prices π_1, π_2 that is determined from (9) and (22) only the data are taken in a slightly different form. The generalization to n sectors is obvious.

This means that if the residual rates ε'_1 and ε'_2 as defined by (20) are constant, we can reason about the volume figures in Table 3 very much in the same way as we can about the strictly physical quantities.

To be more precise: In Table 3 there are 12 variables connected by the 12 equations

$$\begin{aligned} X_1 &= X_{21} + W_1 + B_1 + \varepsilon_1 = X_{12} + X_{1x} \\ X_2 &= X_{12} + W_2 + B_2 + \varepsilon_2 = X_{21} + X_{2x} \end{aligned} \tag{23}$$

and (16)—(17) and (20) where the coefficients X'_{kh}, W'_h, B'_h and ε'_h are constants (and hence may be determined by observing the content of Table 3 in a base year). The first two equations in (23) reduce to conditions on the coefficients. Hence two degrees of freedom in the variables. This checks with the remarks in connection with Table 3.

The two degrees of freedom that remain in Table 3 under the conditions specified — constant physical coefficients, constant

factor prices and constant residual rates — may be generated by letting X_1 and X_2 vary. Or we may use X_{1*} and X_{2*} as basis variables and use other equations to express X_1 and X_2.

D. *Constant Coefficients in the Volume Sense Entail Constant Residual Rates*

If we assume a model of Table 3 where (16)—(17) hold with constant coefficients, we can conclude that the residual rates ε'_k defined by (20) are constants.

Indeed, introducing into the left hand equations in (23) — which follow from the balancing principles of Table 3 — the expressions for X_k, X_{kh}, W_h, B_h from (16)—(17), we get

$$\begin{aligned} \varepsilon'_1 &= 1 - (X'_{21} + W'_1 + B'_1) \\ \varepsilon'_2 &= 1 - (X'_{12} + W'_2 + B'_2). \end{aligned} \tag{24}$$

Hence: If X'_{kh}, W'_h, B'_h are constants, ε'_1 and ε'_2 must also be constants. We are thus back to the same type of analysis as was discussed under subsection *C*.

Having reduced in this way the whole formulation to the figures contained in Table 3, we may *drop* the assumption of an underlying strictly physical structure which we started from, and simply reason about the figures of Table 3 as *value figures reckoned at base year prices*. This formulation will apply even though there is a great variety of individual goods that enter into each aggregate X_k or X_{kh} etc. For all practical purposes these figures could be interpreted as volume indices. And it would seem plausible in many cases to make the assumption of constant input-output coefficients reckoned in such figures.

If we take the volume figures as the basis of the analysis, the product prices π_1, π_2 become indetermined, and the same is true of the factor prices π_w, π_b. Indeed, if in (22) we insert for x'_{kh} from (18), and similarly use (19) we simply get back to (24). The product prices and factor prices can now simply be looked upon as conventional multipliers by which we define "the strictly physical quantities" in (13)—(15). If the "strictly physical quantities" are well defined and observable, we can, of course, deduce the factor prices π_w, π_b and product prices π_1, π_2 that must prevail in order that we shall get the observed

volume figures (in base prices year prices) X_1, X_2, W_1, W_2 etc.

E. *The Aggregate Residual Rates*

The residual rates ε_h' express the input of residual substance that is made *directly* into sector h, reckoned per unit of total output X_h from sector h. We can also consider the *aggregate* residual rates $\varepsilon_h^{\cdot}$ defined by

(25) $\varepsilon_h^{\cdot} =$ that part of X_h which is due to the input of residual substance in any sector, assuming that all residual substance is everywhere passed on to other sectors or to final output in the same proportion as the *volume* of cross deliveries or the final deliveries. In other words all units of output from a given sector contains the same amount of residual substance.

When the aggregate residual rates are defined in this way, they must satisfy the equations

$$\begin{aligned} \varepsilon_1^{\cdot} - \varepsilon_2^{\cdot} X_{21}' &= \varepsilon_1' \\ -\varepsilon_1^{\cdot} X_{12}' + \varepsilon_2^{\cdot} &= \varepsilon_2'. \end{aligned} \tag{26}$$

The first equation in (26) is obtained by noticing that the total outflow of residual substance from sector 1 is $\varepsilon_1^{\cdot} X_1$. This must be equal to the total inflow of residual substance into sector 1, namely the residual substance entered *directly* into sector 1 — this is $\varepsilon_1' X_1$ — *plus* the residual substance that is entered into sector 1 through X_{21} — this is $\varepsilon_2^{\cdot} X_{21}$ —. Dividing this equality by X_1, we get the first equation (26). Similarly for the second equation in (26).

The inputs of labour W_h and imports B_h are *not* to be entered in the above account as they are by definition *not* residual elements. But we could have singled out, say W_k, and considered the *direct* coefficient W_h' as distinct from the *aggregate* coefficient $W_h^{\cdot}$. The reasoning would be the same as in (26).

The generalization of (26) to n sectors is obvious, namely

$$\sum_{k=1}^{n} \varepsilon_k^{\cdot} (\delta - X')_{kh} = \varepsilon_h' \qquad (h = 1, 2, \dots n). \tag{27}$$

The solution of this is

$$(28) \qquad \varepsilon^{\cdot}_k = \sum_{h=1}^{n} \varepsilon'_h (\delta - X')^{-1}_{hk} \quad (k = 1,\ 2, \ldots n).$$

If we take the volume figure coefficients X'_{kh} etc. as *data*, the meaning of the matrix in (27) is clear. If on the other hand we go back for a moment to the interpretation in terms of strictly physical quantities and with constant physical coefficients and constant factor prices, we must remember that the volume figure coefficients X'_{kh} in (27) *depend* on the ε'_h. The volume figure coefficients X'_{kh} will indeed in this case have to be looked upon as determined by (18) where the π_k are given by (22), and hence depend on the ε'_h. This means that if we fall back on the constancy of strictly physical coefficients, we cannot determine the $\varepsilon^{\cdot}_k$ for different ε'_h by retaining the left member matrix in (27) and just changing the right member vector ε'_h. Both the matrix and the vector will have to be changed. On the other hand, if volume figure coefficients are taken as given, we cannot change the ε'_h but must let these magnitudes be determined by (24). The equations (26) have therefore no use for determining the residual rates. The only purpose of the equations is to pass from the *direct* rates ε'_h to the *aggregate* rates $\varepsilon^{\cdot}_h$ or vice versa.

It is a fundamental proposition in input-output theory that equations of the form (26) will have non negative solutions $\varepsilon^{\cdot}_h$ if the ε'_h are non negative.

If we multiply the first equations in (26) by X_1 and the second by X_2 and add the equations, we get, using the equality between the left and right members of (23)

$$(29a) \qquad \varepsilon^{\cdot}_1 X_{1*} + \varepsilon^{\cdot}_2 X_{2*} = \varepsilon_1 + \varepsilon_2.$$

That is to say the total residual substance contained in the final delivery is equal to the total residual substance put into the system.

Since the $\varepsilon^{\cdot}_h$ are non negative (and at least one of them positive if at least one of the ε_h are positive), we see that the total residual substance contained in the sectors products must be larger than the total residual substance put into the system, i.e.

$$(29b) \qquad \varepsilon^{\cdot}_1 X_1 + \varepsilon^{\cdot}_1 X_2 > \varepsilon_1 + \varepsilon_2.$$

This double counting which prima facia appears a little puzzling,

is easily explained: The global product $X_1 + X_2$ has itself emerged after some double counting. In $X_1 + X_2$ is indeed included not only the total primary and direct residual input, but also all *crossdeliveries*. This follows by taking the sum of the left hand equations in (23), which gives

$$(30) \quad X_1 + X_2 = (W_1 + W_2) + (B_1 + B_2) + (\varepsilon_1 + \varepsilon_2) + (X_{12} + X_{21})$$

That is to say the sum of all sector products will increase if we split the sectors further up.

The double counting is only in the total sector products, not in the aggregate residual rates $\dot{\varepsilon}_1$, $\dot{\varepsilon}_2$ as is seen from (29a).

For clear thinking in the variety of situations that arise according to the various systems of assumptions adopted it is essential to be very careful in the notation. It is indeed safe to be so explicit as nearly to appear pedantic.

We will from now on let the symbols used in subsections C and D, i.e. X_k, X_{kh}, X'_{kh} etc. and the corresponding coefficients be strictly interpreted as the *volume figures* and volume figure coefficients, that appear when the residual rates ε'_h are constants and *have a specific set of values.*

These volume figures themselves are recorded in Table 3 and the corresponding coefficients are defined in (16)—(17). With given and constant coefficients the degrees of freedom in this model is, as already stated, equal to n, the number of sections, i.e., in Table 3 it is equal to 2.

F. *The Structure in Semi-Volume Figures*

An essentially new situation arises if we drop the assumption of constant residual rates. We can discuss this situation by going back to the structure expressed in strictly physical terms. We assume constant technical coefficients in this physical structure and also constant factor prices, but the direct residual rates may now be changing. And as they change, they will produce changing product prices and hence changing value figures. These value figures we will term the *semi-volume* figures. In this way of thinking there are $2n$ degrees of freedom, represented, say, by the n physical quantities and the n residual rates.

Instead of discussing the semi-volume structure by the help

of the strictly physical quantities, we can also do it through the residual rates and the *volume* figures as they were defined under subheadings B—E, see in particular the comments in the last part of subsection E.

All the semi-volume magnitudes will be denoted by the superscript sem (standing for semi-volume).

The new situation will be described by a table similar to Table 2 namely by Table 4 and it is through the balancing equations of this new table that the product price concept gets a meaning.

Using an interpretation in terms of the strictly physical structure, we are particularly interested in the connection between the π_k^{sem} and $\varepsilon_1^{\text{sem}}$ and $\varepsilon_2^{\text{sem}}$.

We put up the following definitions, which are similar to (13)—(15).

$$(31) \qquad X_k^{\text{sem}} = \pi_k^{\text{sem}} x_k \qquad X_{k*}^{\text{sem}} = \pi_k^{\text{sem}} x_{k*}$$

$$(32) \qquad X_{kh}^{\text{sem}} = \pi_k^{\text{sem}} x_{kh}$$

$$(33) \qquad W_h^{\text{sem}} = \pi_w^{\text{sem}} w_h \qquad B_{h*}^{\text{sem}} = \pi_{b*}^{\text{sem}} b_h.$$

The semi-volume figures X_k^{sem}, X_{kh}^{sem} etc. measure the production levels, the cross deliveries etc. when the residual rates are chosen as $\varepsilon_k'^{\text{sem}}$ instead of the ε_k' that are associated with the measurement of the volume figures X_k, X_{kh} etc.

In general we will assume

$$(34) \qquad \pi_w^{\text{sem}} = \pi_w \qquad \pi_b^{\text{sem}} = \pi_b$$

but for the symmetry of the formulae we may retain the notation π_w^{sem} and π_b^{sem}. Instead of Table 3 we now get Table 4.

It should be understood that Tables 2, 3 and 4 — as well as a table similar to Table 2 with sem added as superscript on the π and ε — exist at the same time.

In the complete system now considered we again have $2n$ degrees of freedom which may be unfolded by, say, the X_k and the $\varepsilon_k'^{\text{sem}}$. In the case of 2 sectors, there will be 4 degrees of freedom. In the strictly physical system we also had 4 degrees of freedom, when the factor prices π_w and π_b as well as the 6 production coefficients in (8) were given.

If we lean on the interpretation in strictly physical quantities,

it is easy to indicate what the semi-volume figures will be in terms of the volume figures X_k, X_{kh} etc.

TABLE 4

Interflow Table of Semi-Volume Figures Reckoned under the Prices that Prevail when Factor Prices are Constant and Residual Rates are Arbitrarily Given

		Receiving sector No. $h = 1$	Receiving sector No. 2	Final deliveries	Total deliveries
Delivering sector No.	$k = 1$	0	X_{12}^{sem}	X_{1*}^{sem}	X_1^{sem}
	2	X_{21}^{sem}	0	X_{2*}^{sem}	X_2^{sem}
Primary input	Labour	W_1^{sem}	W_2^{sem}	$-(W_1^{\text{sem}}+W_2^{\text{sem}})$	0
	Non competitive imports	B_1^{sem}	B_2^{sem}	$-(B_1^{\text{sem}}+B_2^{\text{sem}})$	0
Residual input		$\varepsilon_1^{\text{sem}}$	$\varepsilon_2^{\text{sem}}$	$-(\varepsilon_1^{\text{sem}}+\varepsilon_2^{\text{sem}})$	0
Grand total		X_1^{sem}	X_2^{sem}	0	

Indeed, adding the superscript sem for the price elements in (22) (except for the factor prices, which are the same) and using (31)—(33), we get

(35)
$$\begin{aligned} \pi_1^{\text{sem}}(1-\varepsilon_1'^{\text{sem}}) - \pi_2^{\text{sem}}x_{21}' &= \pi_w w_1' + \pi_b b_1' \\ -\pi_1^{\text{sem}}x_{12}' + \pi_2^{\text{sem}}(1-\varepsilon_2'^{\text{sem}}) &= \pi_w w_2' + \pi_b b_2' \end{aligned}$$

where

(36)
$$\varepsilon_k'^{\text{sem}} = \frac{\varepsilon_k^{\text{sem}}}{X_k^{\text{sem}}}.$$

Similarly we get

(37)
$$\begin{aligned} \pi_1^{\text{sem}} - \pi_2^{\text{sem}}x_{21}' &= \pi_w w_1' + \pi_b b_1' + \bar{\bar{\varepsilon}}_1^{\text{sem}} \\ -\pi_1^{\text{sem}}x_{12}' + \pi_2^{\text{sem}} &= \pi_w w_2' + \pi_b b_2' + \bar{\bar{\varepsilon}}_2^{\text{sem}} \end{aligned}$$

where

(38)
$$\bar{\bar{\varepsilon}}_k^{\text{sem}} = \frac{\varepsilon_k^{\text{sem}}}{x_k}$$

i.e.

$$(39) \qquad \varepsilon_k'^{\text{sem}} = \frac{\bar{\bar{\varepsilon}}_k^{\text{sem}}}{\pi_k}.$$

We may look upon the two sets of prices π_k and π_k^{sem} simply as special values assumed by the product price functions for different values of the residual rates considered as arguments in these functions.

Through (13)—(15) and (31)—(34) we get, remembering that the strictly physical quantities x_k, x_{hk} etc. are independent of how residual rates are chosen

$$(40) \qquad \frac{X_k^{\text{sem}}}{X_k} = \frac{\pi_k^{\text{sem}}}{\pi_k} \quad \frac{X_{kx}^{\text{sem}}}{X_{kx}} = \frac{\pi_k^{\text{sem}}}{\pi_k}$$

$$(41) \qquad \frac{X_{kh}^{\text{sem}}}{X_{kh}} = \frac{\pi_k^{\text{sem}}}{\pi_k}$$

$$(42) \qquad \frac{W_k^{\text{sem}}}{W_k} = 1 \qquad \frac{B_h^{\text{sem}}}{B_h} = 1.$$

That is to say, we have

$$(43) \qquad X_k^{\text{sem}} = p_k^{\text{sem}} X_k \quad X_{k*} = p_k^{\text{sem}} X_{k*}^{\text{sem}}$$

$$(44) \qquad X_{kh}^{\text{sem}} = p_k^{\text{sem}} X_{kh}$$

$$(45) \qquad W_k^{\text{sem}} = W_k \qquad B_h^{\text{sem}} = B_h$$

where

$$(46) \qquad p_k^{\text{sem}} = \frac{\pi_k^{\text{sem}}}{\pi_k}$$

the p_k^{sem} are *index numbers* of prices, with the base situation chosen as the situation in relation to which the volume figures are defined. These index numbers have a meaning even if no strictly physical quantities are defined. For $p_k^{\text{sem}} = 1$ the semi-volume figures are equal to the volume figures.

The number $2n$ of degrees of freedom is not changed by introducing the semi-volume figures through (31)—(34) or by introducing the price indices through (46). Indeed, to each new magnitude introduced corresponds a definitional equation.

To unfold the $2n$ degrees of freedom, we may use the volume figures X_k and the residual rates $\varepsilon_k'^{\text{sem}}$ defined by (36) (the ε_k' are

fixed as mentioned above, see in particular the comments to (13)—(15)). Instead we may use as basis variables the X_k and the p_k^{sem}. Or the X_k^{sem} and the X_k. Or some other linearly independent set of $2n$ of the variables entering into the complete set up.

From (35) we get by (46), (18) and (19)

$$\begin{aligned} p_1^{\text{sem}}(1-\varepsilon_1'^{\text{sem}}) - p_2^{\text{sem}} X_{21}' &= W_1' + B_1' \\ -\, p_1^{\text{sem}} X_{12}' + p_2^{\text{sem}}(1-\varepsilon_2'^{\text{sem}}) &= W_2' + B_2'. \end{aligned} \tag{47}$$

The coefficient X_{12}', X_{21}' etc. in (47) are determined by (16)—(17) applied in the base year situation. The p_k^{sem} are therefore well defined as functions of the $\varepsilon_k'^{\text{sem}}$. The generalization to n sectors is obvious.

We could also have considered the semi-volume residual rates in the form

$$\bar{\varepsilon}_k^{\text{sem}} = \frac{\varepsilon_k^{\text{sem}}}{X_k}. \tag{48}$$

By (36) and (40) this is the same as

$$\bar{\varepsilon}_k^{\text{sem}} = p_k^{\text{sem}} \varepsilon_k'^{\text{sem}}. \tag{49}$$

With the $\bar{\varepsilon}_k^{\text{sem}}$ given (47) takes the form

$$\begin{aligned} p_1^{\text{sem}} - p_2^{\text{sem}} X_{21}' &= W_1' + B_1' + \bar{\varepsilon}_1^{\text{sem}} \\ -p_1^{\text{sem}} X_{12}' + p_2^{\text{sem}} &= W_2' + B_2' + \bar{\varepsilon}_2^{\text{sem}}. \end{aligned} \tag{50}$$

Note the analogy — and also the difference — between the equations (47) and (50) on one hand and on the other the equations (22) and (9), and also the equations (35) and (37).

For $\bar{\varepsilon}_k^{\text{sem}} = \varepsilon_k'$ we should by (46) get $p_1^{\text{sem}} = p_2^{\text{sem}} = 1$. That this is in fact so, is seen by inserting these values for the p_k^{sem} and comparing with (24).

The solution of (50) is

$$\begin{aligned} p_1^{\text{sem}} &= \frac{W_1' + W_2' X_{21}' + (B_1' + B_2' X_{21}') + (\bar{\varepsilon}_1^{\text{sem}} + \bar{\varepsilon}_2^{\text{sem}} X_{21}')}{1 - X_{12}' X_{21}'} \\ p_2^{\text{sem}} &= \frac{(W_1' X_{12}' + W_2') + (B_1' X_{12}' + B_2') + (\bar{\varepsilon}_1^{\text{sem}} X_{12}' + \bar{\varepsilon}_2^{\text{sem}})}{1 - X_{12}' X_{21}'}. \end{aligned} \tag{51}$$

Again the generalization to n sectors is obvious. Instead of (51) we get

$$(52) \qquad \sum_{k=1}^{n} p_k^{\text{sem}}(\delta - X')_{kh} = W'_h + B'_h + \bar{\varepsilon}_h^{\text{sem}} \quad (h = 1,\ 2 \ldots n).$$

The solution of this is

$$(53) \qquad p_k^{\text{sem}} = \sum_{h=1}^{n} (W'_h + B'_h + \bar{\varepsilon}_h^{\text{sem}})(\delta - X')^{-1}_{hk} \quad (k = 1,\ 2 \ldots n).$$

The last formula suggests immediately the following three component parts of the price p_k^{sem}

$$W^{\cdot}_k = \sum_{h=1}^{n} W'_h(\delta - X')^{-1}_{hk} \text{ due to labour input anywhere in the system.}$$

$$(54) \qquad B^{\cdot}_k = \sum_{h=1}^{n} B'_h(\delta - X')^{-1}_{hk} \text{ due to imports anywhere in the system.}$$

$$\varepsilon_k^{\cdot\text{sem}} = \sum_{h=1}^{n} \bar{\varepsilon}_h^{\text{sem}}(\delta - X')^{-1}_{hk} \text{ due to residual input anywhere in the system.}$$

The last expression in (54) is *aggregate* residual substance in X_k reckoned per unit of X_k. It satisfies the equations

$$(55) \qquad \begin{aligned} \varepsilon_1^{\cdot\text{sem}} - \varepsilon_2^{\cdot\text{sem}} X'_{21} &= \bar{\varepsilon}_1^{\text{sem}} \\ -\varepsilon_1^{\cdot\text{sem}} X'^{\text{sem}}_{12} + \varepsilon_2^{\cdot\text{sem}} &= \bar{\varepsilon}_2^{\text{sem}} \end{aligned}$$

where as before $\bar{\varepsilon}_k^{\text{sem}}$ is direct residual input reckoned per unit of X_k. These equations are analogous to (26). In both cases the residual substance is reckoned per unit of the sector product measured in volume figures. In n sectors (55) is written

$$(56) \qquad \sum_{k=1}^{n} \varepsilon_k^{\cdot\text{sem}}(\delta - X')_{kh} = \bar{\varepsilon}_h^{\text{sem}} \quad (h = 1,\ 2 \ldots n).$$

G. *The Semi-Volume Coefficients and a Modified Definition of the Sector Products*

In analogy with (16)—(17) let semi-volume coefficients be defined by

$$(57) \qquad X'^{\text{sem}}_{kh} = \frac{X^{\text{sem}}_{kh}}{X^{\text{sem}}_h}$$

$$(58) \qquad W'^{\text{sem}}_h = \frac{W_h}{X^{\text{sem}}_h} \quad B'^{\text{sem}}_h = \frac{B_h}{X^{\text{sem}}_h}.$$

Note in this connection (42).

Inserting from (43)—(44) into (57)—(58), we get

$$X'^{\text{sem}}_{kh} = \frac{p^{\text{sem}}_k}{p^{\text{sem}}_h} X'_{kh} \text{ i.e. } X'^{\text{sem}}_{kh} = \frac{p^{\text{sem}}_k X_{kh}}{p^{\text{sem}}_h X_h} \tag{59}$$

$$W'^{\text{sem}}_h = \frac{1}{p^{\text{sem}}_h} W'_h \text{ and } B'^{\text{sem}}_h = \frac{1}{p^{\text{sem}}_h} B'_h. \tag{60}$$

This shows that if the volume coefficients X'_{kh} are constant, the semi-volume coefficients can*not* be constants under changes in residual rates, because, such changes will by (51) make the price indices p^{sem}_k change. There would, of course, be no logical inconsistency in assuming the *semi*-volume coefficients constant, but then the volume coefficients would change under the changes in residual rates.

The real question at issue is to know which is the most *realistic* assumption.

If we assume such a market organization and such a technological structure that an increase in the price of a product will cause an equally large relative decline in its use for cross delivery as well as for total delivery, then the semi-volume coefficient would be constant while the volume coefficient would change. This is seen from (59)—(60).[1])

But if we can assume fixed coefficients in the strictly physical structure, then the volume coefficients must be constant. In what follows I will assume constant volume coefficients.

Then a second question arises: Can we modify the definition of sector product in such a way as to compensate for the variability in semi-volume coefficients?

An obvious answer is that if the *volume* coefficients X'_{kh} are *known* and also the residual rates — either in the form $\varepsilon'^{\text{sem}}_k$ or in the form $\bar{\varepsilon}^{\text{sem}}_k$ — the price indices p^{sem}_k will follow by (47) or (50). Hence we can always by (43)—(44) compute "compensated" variables — namely the X_k and the X_{kh} — which are such that they will be connected by the constant volume coefficients. This procedure is however highly *non linear* and it does not seem

[1]) How realistic such a case would be, is another question.

very promising to proceed to a study of the semi-volume variables along such lines.[2])

Starting from the concepts of semi-volume figures it is, however, possible to introduce certain *modified* definitions of sector products and cross deliveries which are such that they are *approximately* related through the constant volume coefficients.

To arrive at such a formulation we will first rewrite the expressions (51) in the forms

$$(61)\qquad \begin{aligned} p_1^{\text{sem}} &= 1 + \frac{(\bar{\varepsilon}_1^{\text{sem}} - \varepsilon_1') + (\bar{\varepsilon}_2^{\text{sem}} - \varepsilon_2')X_{21}'}{1 - X_{12}'X_{21}'} \\ p_2^{\text{sem}} &= 1 + \frac{(\bar{\varepsilon}_1^{\text{sem}} - \varepsilon_1')X_{12}' + (\bar{\varepsilon}_2^{\text{sem}} - \varepsilon_2')}{1 - X_{12}'X_{21}'}. \end{aligned}$$

The first of these equations follows by writing the numerator in the first equation of (51) in the form

$$(62)\qquad (W_1'+B_1'+\varepsilon_1')+X_{21}'(W_2'+B_2'+\varepsilon_2') + (\bar{\varepsilon}_1^{\text{sem}}-\varepsilon_1')+X_{21}'(\bar{\varepsilon}_2^{\text{sem}}-\varepsilon_2').$$

The first and second parenthesis here are respectively $(1-X_{21}')$ and $(1 - X_{12}')$ by (24). This part of (62) therefore becomes $(1 - X_{21}') + X_{21}'(1 - X_{12}') = 1 - X_{12}'X_{21}'$. This establishes the first equation in (61). Similarly for the second equation in (61).

As a check on (61) we see that p_1^{sem} and p_2^{sem} reduce to 1 if $\bar{\varepsilon}_1^{\text{sem}} = \varepsilon_1'$ and $\bar{\varepsilon}_2^{\text{sem}} = \varepsilon_2'$.

In the case of n sectors, we have

$$(63)\qquad p_k^{\text{sem}} = 1 + \sum_{h=1}^{n} (\bar{\varepsilon}_h^{\text{sem}} - \varepsilon_h')(\delta - X')_{hk}^{-1} \quad (k = 1, 2 \dots n).$$

In the regular case the coefficient of $(\bar{\varepsilon}_1^{\text{sem}} - \varepsilon_1')$ in the first equation of (61), namely $1/1 - X_{12}'X_{21}'$ (in general: the diagonal element $(\delta - X')_{kk}^{-1}$) will be slightly above unity, while the coefficient of $(\bar{\varepsilon}_2^{\text{sem}} - \varepsilon_2')$ in the first equation of (61) will be small since it is multiplied by the coefficient X_{21}', and may therefore be neglected in a first approximation. Similarly in the second equation in (61). That is, we have

$$(64)\qquad p_k^{\text{sem}} = 1+\bar{\varepsilon}_k^{\text{sem}}-\varepsilon_k' \quad \text{(approximately)} \quad (k = 1, 2 \dots n).$$

[2]) The prices are by (50) linear in the $\bar{\varepsilon}_k^{\text{sem}}$, but by (47) non linear in the $\varepsilon_k'^{\text{sem}}$. In any case the deflation by the prices makes the set up non linear.

In other words, as a first approximation the price p_k^{sem} depends only on the residual rate $\bar{\varepsilon}_k$ and not on the other residual rates. And the relation is a simple addition. The "dependency" we speak of now is an (approximate) accounting dependency which hold good regardless of behaviouristic relations.[1])

Multiplying (68) by X_k, we get

$$(65) \quad X_k^{\text{sem}} - \varepsilon_k^{\text{sem}} = X_k(1-\varepsilon_k') \text{ (approximately) } (k = 1, 2, \ldots n).$$

The input-output coefficient in semi volume figures, i.e. $X_{kh}'^{\text{sem}}$ as defined by (57) is not constant. There is, as is seen from the left hand expression in (59) a correction to be applied in the numerator as well as in the denominator in order to reach something that is constant. The correction in the *denominator* can be done with the approximation (65) simply by using the left hand expression in (65) to measure the sector product instead of X_k^{sem}.

We will first consider the case where we make this denominator correction without making the numerator correction. Is this a sound procedure?

By analogy consider the difference $(x_1 - x_2)$ between two stochastic variables. The variance of this difference will be equal to $\text{var. } x_1 + \text{var. } x_2 - 2r\sqrt{\text{var. } x_1 \cdot \text{var. } x_2}$ where r is the correlation coefficient. This expression is larger than var. x_1 if, and only if $\sqrt{\text{var. } x_2/\text{var. } x_1} > 2r$. Therefore, if we know that var. x_2 is *appreciably* larger than var. x_1, it will pay to correct x_2 — that is making it non stochastic — even if we do not correct x_1. And this will apply regardless of the nature of the correlation, whether positive or negative.

In our case the question is if we shall correct for p_h^{sem} in the denominator of the expression to the right in (59) even if we do not correct for p_k^{sem} in the numerator. We know that a change in p_h^{sem} will produce a change in p_k^{sem} in the *same* direction (positive correlation), but the change in p_k^{sem} will be proportionally much smaller if there are many highly intertwined sectors. Hence we

[1]) It is the equation itself, i.e. (64) — or more exactly (61) — which has accounting character. This, of course, does not prevent one or more of the variables from entering into some other relations that are behaviouristic. The expressions "accounting" vs. "behaviouristic" can be used about a relation, not about a variable.

ought to get a more correct result by correcting for p_h^{sem} even if we do not do it for p_k^{sem}.

The above argument is particularly adapted to the case where there is a change in the residual rate in a single sector. To some extent a similar reasoning can be applied in succession to any of the sectors. In each step the correction contemplated will be better than nothing. But it is quite clear that if all residuals change simultaneously, there may occur cases where it would have been better to make no corrections at all in the variables.

For instance if all $\varepsilon_k'^{\text{sem}}$ are equal — i.e. $\varepsilon_k'^{\text{sem}}$ independent of k — we see from (72) that we obtain a better approximation by not making any corrections on the variables, because in this case X_{kh}^{sem} is a constant times X_h^{sem}. The constant is equal to $(1 - \varepsilon_k'/1 - \varepsilon_h')X_{kh}'$.

On the other hand if $(\bar{\varepsilon}_k^{\text{sem}} - \varepsilon_k')$ is independent of k, and hence by (64), p_k^{sem} independent of k, we see from (57)—(59) that X_{kh}^{sem} is again a constant times X_h, but now the constant is simply X_{kh}'.

These cases where the semi volumes figures themselves are connected by constant coefficients are, however, very special. They resemble the case where the residual rates are constant and we get the volume figures.

If we want an approximation that holds — at least roughly — for *any* changes in residual rates — in particular for changes with a small covariance between the individual residual rates — the correction of the denominator to the right in (59) — which leads to (68) — seems to be a workable formula.

The correction for p_h^{sem} can be achieved simply by starting from the exact relation

$$(66) \qquad X_{kh}^{\text{sem}} = p_k^{\text{sem}} X_{kh}' X_h$$

and introducing here the expression for X_h taken from (65). This gives

$$(67) \qquad X_{kh}^{\text{sem}} = p_k^{\text{sem}} \left[\frac{X_{kh}'}{1 - \varepsilon_h'}\right] (X_h^{\text{sem}} - \varepsilon_h^{\text{sem}}) \quad \text{(approximately)}.$$

Dropping at this stage the correction p_k^{sem}, we can write

$$(68) \qquad X_{kh}^{\text{sem}} = \left[\frac{X'_{kh}}{1 - \varepsilon'_h}\right] (X_h^{\text{sem}} - \varepsilon_h^{\text{sem}}) \quad \text{(approximately).}$$

The expression in bracket is a constant and can be determined from the data in the base year. (If the sector product has been defined as $(X_h - \varepsilon_h)$ already in the base year where the coefficients were determined, the value of the bracket will emerge directly.)

This procedure, while rough has the great advantage that it *keeps the model linear,* and it will as a rule — compare the discussion above — at least be better than simply to put

$$(69) \qquad X_{kh}^{\text{sem}} = X'_{kh} X_h^{\text{sem}} \quad \text{(incorrect)}$$

in a case where the residual rates do *not* remain constant.

It is possible to make the first order correction also for the factor p_k in the numerator to the right in (59) but then the model does not remain linear. Indeed we have

$$(70) \qquad p_k^{\text{sem}} = \frac{X_k^{\text{sem}}}{X_k} \quad \text{(exactly).}$$

Introducing here for X_k from (65), we get

$$(71) \qquad p_k^{\text{sem}} = \frac{X_k^{\text{sem}}}{X_k^{\text{sem}} - \varepsilon_k^{\text{sem}}} (1 - \varepsilon'_k) \quad \text{(approximately).}$$

And inserting this in (67), we get [1])

$$(72) \qquad X_{kh}^{\text{sem}} = \frac{X_k^{\text{sem}}}{X_k^{\text{sem}} - \varepsilon_k^{\text{sem}}} \left[\frac{(1 - \varepsilon'_k) X'_{kh}}{1 - \varepsilon'_h}\right] (X_h^{\text{sem}} - \varepsilon_h^{\text{sem}}) \quad \text{(approximately).}$$

H. *The Formulation in Non-Residual Cost*

Let us introduce *the aggregate non-residual part* of the price of the output from sector k. This is that part of p_k^{sem} which is due to the input of primary factors (in Table 4) labour and imports). This part of p_k^{sem} is given as the first two terms in (54) which — in the case $n = 2$ — are given by the terms in

[1]) Dividing by the first fraction in (72), the left member becomes $X_{kh}^{\text{sem}}(1 - \varepsilon_k'^{\text{sem}})$ If this is taken as definition of a *corrected* cross delivery, we get a relation with constant coefficient. But this relation is not linear in X_{kh}^{sem}, X_h^{sem} and $\varepsilon_k^{\text{sem}}$.

(51) that do not depend on $\bar{\varepsilon}_1^{\text{sem}}$ and $\bar{\varepsilon}_2^{\text{sem}}$.

We denote this part

$$(73) \quad p_k^{\text{nori}} = \sum_{k=1}^{n} (W'_h + B'_h)(\delta - X')^{-1}_{hk} \quad (k = 1, 2 \ldots n).$$

As long as the production coefficients reckoned in volume figures are constant, the prices (73) are constant. The corresponding non residual parts of the sector products, cross deliveries and final deliveries are

$$(74) \qquad X_k^{\text{nori}} = p_k^{\text{nori}} X_k \quad X_{k*}^{\text{nori}} = p_k^{\text{nori}} X_{k*}$$

$$(75) \qquad X_{kh}^{\text{nori}} = p_k^{\text{nori}} X_{kh}$$

$$(76) \qquad W_h^{\text{nori}} = W_h \qquad B_h^{\text{nori}} = B_h.$$

Since these non residual parts of the volume figures are simply *proportional* to the volume figures, nothing is gained by working with these variables instead of the volume figures. Both sets of variables will exactly satisfy relations with constant coefficients, provided the volume structure coefficients are constant. No further attention will therefore be paid to the structure in non residual parts.

I. *Formulation in Factor Costs*

Finally we will consider a breakdown of $\varepsilon_k^{\text{sem}}$ in the two parts

$$(77) \qquad \varepsilon_k^{\text{sem}} = \delta_k^{\text{sem}} + T_k^{\text{sem}} \qquad (k = 1, 2. \ldots n)$$

where δ_k^{sem} stands for profits and T_k^{sem} for taxes. More specifically we may interpret δ_k^{sem} as profits before the deduction of *direct* taxes, so that T_k^{sem} will stand for *indirect* taxes.

We have now n more degrees of freedom, i.e. $3n$ degrees altogether. They may be represented, say, by the X_k, and the rates $\bar{\varepsilon}_k^{\text{sem}}$ and $\bar{T}_k^{\text{sem}}$, where $\bar{\varepsilon}_k^{\text{sem}}$ is defined by (48) and

$$(78) \qquad \bar{\delta}_k^{\text{sem}} = \frac{\delta_k^{\text{sem}}}{X_k} \qquad \bar{T}_k^{\text{sem}} = \frac{T_k^{\text{sem}}}{X_k}$$

so that

$$\bar{\varepsilon}_k^{\text{sem}} = \bar{\delta}_k^{\text{sem}} + \bar{T}_k^{\text{sem}}.$$

Assuming that the coefficients in the volume structure are constant, we know that if the $\bar{\varepsilon}_k^{\text{sem}}$ are changed, the p_k^{sem} must

follow in the way previously discussed. This, however, does not say anything about the way in which the prices will change if the $\bar{T}_k^{\text{sem}}$ change. Conceivably any change in the $\bar{T}_k^{\text{sem}}$ may be compensated by opposite changes in the $\bar{\delta}_k^{\text{sem}}$ so that the $\bar{\varepsilon}_k^{\text{sem}}$ remain constant and hence the p_k^{sem} constant. Or a smaller or larger part of the change in $\bar{T}_k^{\text{sem}}$ may be absorbed in the prices.

In a market of a more or less conventional sort it is perhaps plausible to assume, as a very simple case, that the $\bar{T}_k^{\text{sem}}$ will affect the prices *directly and fully* in the sense that we have

(79) $p_k^{\text{sem}} = 1 + \bar{T}_k^{\text{sem}} - T'_k$ (approximately) $(k = 1, 2 \ldots n)$

where

(80) $$T'_k = \frac{T_k}{X_k} = \text{indirect tax rate in the base year.}$$

If this is so, we get by a reasoning analogous to that connected with (64)—(65).

(81) $X_k^{\text{sem}} - T_k^{\text{sem}} = X_k(1 - T'_k)$ (approximately) $k = 1, 2 \ldots n$).

so that in analogy with (71)—(72) we may put

(82) $$X_{kh}^{\text{sem}} = \left[\frac{X'_{kh}}{1 - T'_h}\right] (X_h^{\text{sem}} - T_h^{\text{sem}}) \quad \text{(approximately).}$$

The expression in brackets is a constant to which we may attach comments similar to those connected with (68).

The set up (81) has been used in the Oslo median model.[1])

J. *Formulation in current values*

If we consider also the factor prices as variables, we are led to the concepts $X_k^{\text{cur}} = p_k X_k$, $X_{kh}^{\text{cur}} = p_k X_{kh}$, etc. Note that p_k^{sem} stands for the price concepts that emerge when the factor prices, i.e. the price concepts for W_k and B_k, are *constant*, while p_k stand for the corresponding concepts when the wage rate and the import prices may change. I.e. that emerge when the factor price concept is expressed by a general wage index q (with

[1]) The Oslo median model contained several specifications that go beyond those considered here, but in essence we can say that what is here denoted $(X_h^{\text{sem}} - T_h^{\text{sem}})$ and X_{kh}^{sem} respectively, was there denoted X_h and X_{kh}.

$q = 1$ in the base situation) and the import prices are arbitrarily given. Correspondingly the X_k^{cur} and the X_{kh}^{cur} are current values as they emerge when applying the general price indices p_k.

When the current values are *deflated,* we get back to the volume figures. For any individual sector product this deflation is simply

$$\text{(83)} \qquad \text{defl. } X_k^{\text{cur}} = \frac{X_k^{\text{cur}}}{\text{Pr. ind. } (X_k^{\text{cur}})} = \frac{X_k^{\text{cur}}}{p_k} = X_k.$$

For the global output as a whole we have

$$\text{(84)} \qquad \text{defl. } (X_1^{\text{cur}} + X_2^{\text{cur}}) = \frac{X_1^{\text{cur}} + X_2^{\text{cur}}}{\text{Pr. ind. } (X_1^{\text{cur}} + X_2^{\text{cur}})}\,.$$

If we use a Laspeyre price index, (84) is further reduced to

$$\text{(85)} \qquad \text{defl. } (X_1^{\text{cur}} + X_2^{\text{cur}}) = \frac{X_1^{\text{cur}} + X_2^{\text{cur}}}{\dfrac{p_1 X_1 + p_2 X_2}{X_1 + X_2}} = X_1 + X_2.$$

A similar reduction takes place if we deflate the semi volume figures. That is to say, in order to arrive at a measurement of the global output that is independent of such price effects as may be produced by residual input elements, or more specifically that part of these elements which is represented by indirect taxes, it is not necessary to use measurements as $(X_k^{\text{cur}} - \varepsilon_k^{\text{cur}})$ or $(X_k^{\text{cur}} - T_k^{\text{cur}})$ for the sector products (compare by analogy (65) and (81)). We can use the total value X_k^{sem} and afterwards deflate.

Another aspect of this question is that such differences as $(X_k^{\text{cur}} - \varepsilon_k^{\text{cur}})$ or $(X_k^{\text{cur}} - T_k^{\text{cur}})$ only represent first order corrections. The complete correction is obtained by computing the prices p_1, p_2 by formulae analogous to (51) or (53) — or observing them — and then using these prices for the deflation process.

LAWRENCE R. KLEIN

The Efficiency of Estimation in Econometric Models

At an early stage in the development of mathematical statistics and in its application to economics Professor Hotelling steered our thinking in the right direction towards the relationship between problems of parameter estimation from samples and problems of prediction outside the confines of the sample. The essence of the matter is contained in his 1929 paper written jointly with H. Working.[1]) A more concise presentation was given later in Hotelling's appraisal of a research publication in psychology and sociology.[2]) Hotelling's formulation of the prediction problem will long serve as a model in econometrics. In following up recent contributions on the efficiency of alternative methods of estimation in econometrics, it is found immediately useful to turn to Hotelling's basic formula for the standard error of forecast.

First I shall take up some problems in connection with the use of conventional least-squares methods of estimation in econometrics and then go on more specifically to the problems of prediction which rely so heavily on Hotelling's inspirational work.

A Revival of Support for Least-Squares Estimates

Recent arguments in favor of reversion to conventional application of the method of least squares for estimating parameters in single equations in simultaneous systems dealt with in econometrics have stressed the comparative efficiency of this

[1]) H. Hotelling and H. Working, "Applications of the Theory of Error to the Interpretation of Trends," *Journal of the American Statistical Association*, Vol. 24 (March Supplement, 1929), pp. 73—85.

[2]) H. Hotelling, "Problems of Prediction," *The American Journal of Sociology*, XLVIII (1942), 61—76.

method. For example, in the discussions at the 1954 meetings of the Econometric Society in Uppsala, H. Theil outlined a theorem showing that the generalized variance of least-squares estimates of the parameters in a single equation is at least as small as that of limited-information-maximum-likelihood estimates.[3]) It has long been suggested that it may be rewarding to gain some efficiency at the expense of bias by using the least-squares method, but the formal proof of the superior efficiency has only just been set out by Theil. Of course, we must know more about the "utility function" of the users of estimated models before we can judge about the relative importance of bias and efficiency.

In this paper, I should like to suggest that Theil's results and, in fact, much of the discussion on the relative merits of different methods is misplaced. In systems of equations, with several parameters in each equation, it is misleading to look at individual parameters, one by one, or even restricted groups of them in reaching overall judgments about the importance of bias or efficiency. Users of econometric models are often not really interested in particular structural parameters by themselves. They are interested in the *solution* to the system, under alternative sets of conditions. In other, more technical, words they are interested in the *reduced forms* of the estimated system.[4]) It is the difference between *partial* and *general* analysis that is involved. It is conceivable that partial analysis is an end, in itself, for some problems — possibly those of a purely pedagogical nature — but most problems call for a more complete analysis of the system.

The transformation of a structural system to its reduced form can be associated with the process of *forecasting*. We shall use

[3]) "Report of the Uppsala Meeting, August 2—4, 1954," *Econometrica*, Vol. 23, (April, 1955), pp. 204—5. Limited Information and other methods of estimation in econometric models are described in various publications. A good summary is found in Wm. C. Hood and T. C. Koopmans, "The Estimation of Simultaneous Linear Economic Relationships," *Studies in Econometric Method*, (New York: John Wiley and Sons, 1953).

[4]) Theil points out to me that for treatment of structural change and appraisal of the *a priori* "reasonableness" of parameter estimates one would be primarily interested in the individual structural estimates.

this term in a general sense in this paper, that is in the sense of making estimates of endogenous economic magnitudes outside the realm of past experience. In this sense, a wide variety of problems of empirical economic analysis are forecasting problems.

Propositions valid for partial systems may not carry over when complete systems are studied. Or even propositions valid for *structural* parameters may not be valid for *reduced form parameters*. A beautiful property of the maximum-likelihood method of estimation is that its characteristics are preserved under single-valued transformation of variables. Let Θ be an unknown parameter, $\hat{\Theta}$ its maximum-likelihood estimate, and $f(\Theta)$ a single-valued transformation function. Then it follows that the maximum-likelihood estimate of $f(\Theta)$ is given by $f(\hat{\Theta})$. As a result of this proposition, the desirable features of maximum likelihood estimates of structural parameters remain as desirable features when the parameters and estimates are transformed into reduced form coefficients. An analogous property does not hold for the method of least squares in general.

Bias-Structural Equations and Reduced Forms

First, let us consider the question of bias. By comparing one-by-one the coefficients of a system estimated by the method of least squares with corresponding coefficients estimated by some unbiased method, investigators sometimes conclude superficially that the amounts of bias are unimportant. From studies of numerical methods of solving linear equation systems with parameters subject to error, we learn, however, that small errors in coefficients may lead to sizeable errors in the final solution.

Suppose that we have a linear equation system

(1) $$By_t + \Gamma z_t = u_t, \quad y \text{ jointly dependent, } z \text{ predetermined,}$$

with reduced form

(2) $$y_t = -B^{-1}\Gamma z_t + B^{-1}u_t.$$

We shall denote a set of unbiased or consistent estimates as $\hat{B}$ and $\hat{\Gamma}$. Least-squares estimates will then be written as

$$\hat{B} + D(B), \ \hat{\Gamma} + D(\Gamma).$$

$D(B)$ and $D(\Gamma)$ are discrepancy matrices showing how the least-squares estimates differ from the set $\hat{B}$, $\hat{\Gamma}$. For a given z_t-vector we shall then be interested in the effect of the discrepancies on the solution vector.

A first order approximation to the solution of

$$[\hat{B} + D(B)][\hat{y}_t + D(y_t)] + [\hat{\Gamma} + D(\Gamma)]z_t = 0$$

is given by

$$D(y_t) = -\hat{B}^{-1}[D(\Gamma)z_t + D(B)\hat{y}_t], \tag{3}$$

where

$$\hat{y}_t = -\hat{B}^{-1}\hat{\Gamma}z_t.$$

Except for the possibility that errors in the Γ-matrix, $D(\Gamma)$, compensate errors in the B-matrix, $D(B)$, these original errors will be reflected in errors in the vector, $D(y_t)$ after multiplication by $\hat{B}^{-1}$. In some problems $\hat{B}^{-1}$ can be a large factor, as will be illustrated below in a simple example.

Consider, for illustrative purposes, the simple multiplier model

$$C_t = \alpha Y_t + u_t, \tag{4}$$

$$Y_t = C_t + I_t. \tag{5}$$

C_t = consumption (endogenous)
Y_t = income (endogenous)
I_t = investment (exogenous)
u_t = random disturbance.

The unbiased estimate of the marginal propensity to consume is a, and the biased least-squares estimate is $a + e$.[5]) The bias, when looked at from a partial point of view is simply e, but when considered from the more general point of view of the whole system it is $e/1 - a - e$, as can be seen from

[5]) Haavelmo derives an explicit expression for the bias in this model and finds it positive under fairly general conditions. T. Haavelmo. "Methods of Measuring the Marginal Propensity to Consume," *Journal of the American Statistical Association*, Vol. 42 (1947), pp. 105—122.

$$C_t = (a + e)(C_t + I_t)$$

$$(6) \qquad C_t = \frac{a+e}{1-a-e} I_t = \left[\frac{a}{1-a} + \frac{e}{(1-a)(1-a-e)}\right] I_t$$
$$= \left[a + \frac{e}{1-a-e}\right] \hat{Y}_t,$$

where

$$\hat{Y}_t = \frac{1}{1-a} I_t.$$

In the reduced form, or multiplier, equation the bias is magnified by the factor $1/1 - a - e$, which will be of the order of magnitude of the multiplier.[6]) Another way of looking of the matter is to observe that the percentage bias is smaller in absolute value in the structural equation than in the reduced form equation.

$$\left|\frac{e}{a}\right| < \left|\frac{e}{a(1-a-e)}\right|, \quad \text{as long as } |1 - a - e| < 1.$$

In this simple case, it is clearly seen that a discrepancy in the structural equation e, is magnified by the multiplier, $1/1 - a - e$. The multiplier plays the role of $\hat{B}^{-1}$ in the more general formulation (3), showing the possibility of comparatively small discrepancies becoming comparatively large in the final result.

Least-Squares Efficiency—Structural Equation and Reduced Form

This simple model of the multiplier process is useful in providing an example to show that Theil's proposition about the efficiency of least-squares methods cannot be extended to reduced form parameters. The efficiency properties of least-squares estimation of the parameter in the structural equation are not preserved under transformation to the reduced form, just as we know that the point values of least-squares estimates vary with the direction of minimization of squared residuals.

The variance of the direct least-squares estimate of the marginal

[6]) Haavelmo, *op. cit.*, shows numerically how a difference of .06 in the marginal propensity to consume becomes a difference of .68 in the multiplier.

propensity to consume is given by

$$\text{var } (a + e) = \text{var} \frac{\Sigma C_t Y_t}{\Sigma Y_t^2} = \text{var } \frac{\Sigma(\alpha Y_t + u_t)Y_t}{\Sigma Y_t^2}$$
$$= \text{var}\left(\alpha + \frac{\Sigma u_t Y_t}{\Sigma Y_t^2}\right) = \text{var } \frac{\Sigma u_t Y_t}{\Sigma Y_t^2}.$$

Applying a first order approximation formula of a function of random variables to the right hand expression we get the classical type result.[7])

$$\text{var } (a + e) = \frac{\text{var } (u)}{\Sigma Y_t^2}, \tag{7}$$

in the limit as the sample size $\to \infty$. In deriving this result the standard independence assumptions are used about u_t and Y_t for unequal subscript values. Thus for large samples the classical formula is used even though u_t and Y_t are not independent (for equal subscript values).

In forecasting Y_t from estimates of the reduced form equation

$$Y_t = \frac{1}{1 - \alpha} I_t + \frac{1}{1 - \alpha} u_t,$$

we are interested in the variance of the estimated multiplier

$$\text{est. } \frac{1}{1 - \alpha}.$$

The least-squares estimate of the structural equation leads to an estimate of the multiplier as

$$\text{l.s.est. } \frac{1}{1 - \alpha} = \frac{1}{1 - a - e},$$

while the corresponding unbiased estimate will be written as

$$\text{m.l.est. } \frac{1}{1 - \alpha} = \frac{1}{1 - a} \quad (\text{m.l.} = \text{maximum likelihood}).$$

We approximate the variance of the least-squares estimate as

[7]) This is, of course, the same result that Theil derives by a different method in his comparison of the efficiency of least-squares and limited-information estimates. He deals with the *estimated* variance determined from the sample observations.

$$
\begin{aligned}
(8)\quad & \operatorname{var}\frac{1}{1-a-e}=\frac{\operatorname{var}(a+e)}{(1-a-e)^4}=\frac{\left(\dfrac{1}{1-a-e}\right)^2\operatorname{var}(u)}{(1-a-e)^2\Sigma Y_t^2}\\
&=\frac{\left(\dfrac{1}{1-a-e}\right)^2\operatorname{var}(u)(\Sigma Y_t^2)^2}{(\Sigma Y_t^2-\Sigma C_tY_t)^2\Sigma Y_t^2}=\frac{\operatorname{var}(u)}{(1-a-e)^2}\,\frac{\Sigma Y_t^2}{(\Sigma I_tY_t)^2}.
\end{aligned}
$$

The maximum-likelihood estimator is simply that multiplier value calculated from the least-squares regression of Y_t on I_t,

$$
(9)\qquad \operatorname{var}\left(\frac{1}{1-a}\right)=\frac{\left(\dfrac{1}{1-\alpha}\right)^2\operatorname{var}(u)}{\Sigma I_t^2}.
$$

With a positive bias, $e>0$, we have the inequality

$$
\frac{\operatorname{var} u}{(1-\alpha)^2}\leqq \operatorname{plim}\frac{\operatorname{var}(u)}{(1-a-e)^2}
$$

since $\operatorname{plim} a=\alpha$,
and we assume $0<1-a-e<1$.
Also

$$
\frac{1}{\Sigma I_t^2}\leqq\frac{\Sigma Y_t^2}{(\Sigma I_tY_t)^2};
$$

hence

$$
\operatorname{var}\left(\frac{1}{1-a}\right)\leqq \operatorname{plim}\operatorname{var}\left(\frac{1}{1-a-e}\right).
$$

Thus our example shows clearly that the efficiency properties of least-squares structural estimates do not always carry over to the reduced form equations. This example is, of course, special as well as simple. The maximum-likelihood estimate of the reduced form equation is simultaneously, or can be transformed into, a least-squares estimate of the reduced form, a full-information-maximum-likelihood estimate of the whole system, and a limited-information-maximum-likelihood estimate of the consumption function. All three of these identical estimates are therefore superior to least-squares structural estimates, in a sense, for this model.

If a model is exactly identified, there exists a set of least-squares estimates that provide efficient estimates of the reduced forms. These are also maximum-likelihood estimates of the reduced form and can be transformed into maximum-likelihood estimates of the structural parameters. They are, however, very particular least-squares estimates and not those that are customarily obtained from the separate treatment of each structural equation. The least-squares estimates of the structural equation, in the form usually made, do not in general preserve their efficiency properties under transformation to reduced form parameters. To put this in concrete terms of the simple model considered in this section, we note that the least-squares estimate of the marginal propensity to consume does not provide an efficient estimate of the multiplier (although a least-squares estimate of the multiplier would, in fact, be efficient). In over-identified systems, the lack of invariance of the efficiency property under transformation proves to be more important.

In the next section, we shall compare efficiency of least-squares and full- and limited-information-maximum-likelihood estimates of the reduced forms, in more general cases, including overidentification. In another section, however, we shall present some results of a sampling experiment with small samples and overindentification, which show clearly the failure of least-squares estimates to retain efficiency under transformation.

The Efficiency of A Priori Restrictions

It has frequently been remarked that by imposing *a priori* restrictions on an economic model we gain efficiency of estimation because more information is brought to bear on the problem than is the case in the absence of such restrictions.[8]) It is the purpose of this section to give a formal proof of this point of view and to interpret it.

Liu has posed the seemingly paradoxical proposition: Least-squares estimates of the reduced form parameters in linear systems, ignoring all *a priori* restrictions, lead to a higher point on the likelihood function, assuming normally distributed disturb-

[8]) See e.g., T. C. Koopmans and Wm. C. Hood, *op. cit.*, esp. p. 176.

ances, than do restricted maximum-likelihood estimates; therefore the unrestricted estimates are to be preferred.[9]) Put in another way, he observes that squared discrepancies between predicted and actual values of endogenous variables in a linear model will be smaller over the sample period if calculated from (unrestricted) least-squares estimates of the reduced form equations than if calculated from any set of estimates of structural parameters using *a priori* restrictions.[10]) If the least-squares values of the reduced forms give better "predictions" over the sample period, should we not expect them to give better predictions outside the sample?

While the least-squares estimates of the reduced forms are *consistent* and while they lead to minimal squared residuals; they do not lead to *efficient* estimates of the reduced form parameters. We shall now turn to the proof of the proposition that the more one uses valid information in the form of *a priori* restrictions imposed on the system, the more efficient are the estimates.

Let us write a linear model as

$$By_t + \Gamma z_t = u_t, \tag{1}$$

with reduced form

$$y_t = -B^{-1}\Gamma z_t + B^{-1}u_t. \tag{2}$$

y_t, z_t and u_t are column vectors with n, m, and n components, respectively. B is a square, nonsingular, matrix of order nxn, while Γ is rectangular of order nxm.

We can also write the reduced form equation as

$$y_t = \Pi z_t + v_t, \tag{10}$$

obscuring the relation between its coefficients and those of the

[9]) T. C. Liu, "A Simple Forecasting Model for the U. S. Economy," *International Monetary Fund Staff Papers*, IV (August, 1955) 464—66, Liu's argument is actually more in terms of a supposed inherent tendency towards underidentification in models of an economic system than in terms of the point reached on the likelihood function with and without restriction. He uses the latter point, however, in justification of his claim of lack of identifiability. His general conclusion is that we can do no better than to make unrestricted least-squares estimates of the reduced forms.

[10]) In the case of exact identification, least squares estimates of the reduced form parameters will coincide with fully restricted maximum likelihood estimates.

structural equations. On multiplying both sides of this equation by B, however, we find

$$B\Pi = -\Gamma, \tag{11}$$

by equating coefficients of like variables in the reduced form and structural equations. Some elements of Γ are zero; therefore $B\Pi$ has the same zero elements. These are the restrictions. Let us write, symbolically,

$$(B\Pi)_r = 0 \tag{12}$$

to show that r elements of Γ are zero, and that corresponding elements of $B\Pi$ are zero.

The logarithm of the likelihood function of the entire system can be written as

$$L = \text{const.} - \frac{T}{2}\log|\Omega_v| - \tfrac{1}{2}\sum_{t=1}^{T} v_t'\Omega_v^{-1}v_t + \lambda'(B\Pi)_r. \tag{13}$$

Ω_v = matrix of variances and covariance of elements of v_t. λ is a vector of Lagrange multipliers with r non-zero elements. If there were no restrictions on the system, variances and covariances of est. Π are given by

$$\left\| -\frac{\partial^2 L}{\partial\pi_{ik}\,\partial\pi_{jl}} \right\|^{-1} = \left\| \sigma^{ij} M_{zz} \right\|^{-1} = \left\| \sigma_{ij} M_{zz}^{-1} \right\|. \tag{14}$$

σ^{ij} = typical element of Ω_v^{-1}.

σ_{ij} = typical element of Ω_v.

M_{zz} = moment matrix of predetermined variables, z_t.

The elements of this inverse matrix (14) are the variances and covariances of unrestricted least squares estimates of Π. For any particular equation in this complete set of reduced forms, the appropriate variance-covariance matrix is $\sigma_{ii}M_{zz}^{-1}$. To determine the variance of forecast for any endogenous variable, outside the sample values, we have, by Hotelling's formulation of the problem of prediction error,

$$S_F^2 = \sigma_{ii}(1 + z_F' M_{zz}^{-1} z_F), \tag{15}$$

where z_F denotes a vector of values assigned to predetermined variables in the forecast period.

Suppose now that we have r restrictions on the maximization of the likelihood function, then the variances and covariances of est. Π are given by the N.W. principal minor, $\hat{A}$, on the right hand side of the following expression

$$(16)\qquad \left\| \begin{array}{c:c} -\dfrac{\partial^2 L}{\partial \pi_{ik}\partial \pi_{jl}} & \Lambda \\ \hdashline \Lambda' & 0 \end{array} \right\|^{-1} = \left\| \begin{array}{c:c} \hat{A} & \hat{\Lambda} \\ \hdashline \hat{\Lambda}' & \Phi \end{array} \right\|.$$

The bordering matrix Λ consists of elements of λ obtained as $-\partial^2 L/\partial \pi_{ij}\partial b_{kl}$, where b_{kl} are elements of B.

Theorem: Consider an mxm principal minor of $\hat{A}$ corresponding to the i-th reduced form equation. Denote this principal minor by $\hat{A}_{ip}$.

It follows that

$$z'_F \hat{A}_{ip} z_F \leqq \sigma_{ii}\, z'_F M_{zz}^{-1} z_F.$$

Proof: Denote a vector of mn elements by w.

$$(17)\qquad \text{Let det.} \left\| \begin{array}{c:c} -\dfrac{\partial^2 L}{\partial \pi_{ik}\partial \pi_{jl}} & \lambda \\ \hdashline \lambda' & 0 \end{array} \right\| = \Delta,$$

where λ is a vector.

Form the difference

$$\delta = \frac{\sum\limits_{i,j=1}^{mn} \Delta_{00.ij} w_i w_j}{\Delta_{00}} - \frac{\sum\limits_{i,j=1}^{mn} \Delta_{ij} w_i w_j}{\Delta}.$$

Δ_{00} is formed from Δ by deleting the last row and column. $\Delta_{00.ij}$ is formed from Δ_{00} by deleting the i-th row and j-th column.

$$\delta = \frac{\Delta \sum\limits_{i,j=1}^{mn} \Delta_{00.ij} w_i w_j - \Delta_{00} \sum\limits_{i,j=1}^{mn} \Delta_{ij} w_i w_j}{\Delta \Delta_{00}}$$

By Jacobi's theorem on determinants[11]) we have

[11]) See e.g., A. C. Aitken, *Determinants and Matrices* (London: Oliver and Boyd, 1942), pp. 98—99.

$$\Delta\Delta_{00.ij} = \Delta_{00}\Delta_{ij} - \Delta_{0j}\Delta_{0i};$$

therefore

$$(18) \qquad \delta = -\frac{\sum_{i,j=1}^{mn} \Delta_{0j}\Delta_{0i}w_i w_j}{\Delta\Delta_{00}} = -\frac{\left(\sum_{i=1}^{mn} \Delta_{0i}w_i\right)^2}{\Delta\Delta_{00}} \geqq 0.$$

The difference, δ, is positive since Δ_{00} is a positive definite quadratic form. Determinants of positive definite quadratic forms obtained by successive rows and columns of bordering alternate in sign; hence

$$-\Delta\Delta_{00} > 0.$$

Thus by bordering

$$(19) \qquad \det. \left\| -\frac{\partial^2 L}{\partial\pi_{ik}\partial\pi_{jl}} \right\| = \Delta_{00}$$

with one row and column, we find

$$(20) \qquad w'(\Delta^{-1})_{00}w \leqq w'(\Delta_{00})^{-1}w.$$

By bordering Δ with a row and column to form $({}_{11}\Delta)$, we similarly find

$$(21) \qquad w'[({}_{11}\Delta)^{-1}]_{00.00}w \leqq w'(\Delta^{-1})_{00}w,$$

and so on for successive borderings. By letting all elements of w vanish except the m elements in z_F corresponding to the mxm principal minor of $\hat{A}$ associated with i-th reduced form equation, we have

$$(22) \qquad z'_F\hat{A}_{ip}z_F \leqq \sigma_{ii}z'_F M_{zz}^{-1}z_F,$$

as was to be proved.

The total variance of forecast from a system of structural equations is composed of two factors, the variance of disturbances σ_{ii} and a quadratic form (plus unity) in the assumed values of the predetermined variables. The matrix of this quadratic form when multiplied by σ_{ii} is the variance-covariance matrix of reduced form parameter estimates. By adding information to the system in the form of restrictions on the parameters, we decrease the magnitude of this quadratic form. The *estimated* value of forecast error will reflect differences in estimation

procedure, the calculated variance of residuals being lowest when no restrictions are used. Nevertheless, the underlying or inherent error using the true value of σ_{ii} in the formula for prediction error will be smaller, the more one employs *a priori* restrictions in calculating the structural characteristics.

The succession of inequalities derived shows that full-information-maximum-likelihood estimates will be more efficient than limited-information-maximum-likelihood estimates since more restrictions are used with the former than with the latter set. Both of these methods, in turn, will be more efficient than unrestricted least-squares estimates of the reduced forms.

In connection with limited-information estimates a word of explanation is in order. When any single equation in a system is being estimated, this method will produce a set of reduced form estimates, separately involving each of the endogenous variables in that equation. In so far as a single endogenous variable appears in several different structural equations, there will be several possible reduced form equations that could serve as alternative forecasting equations. The best among these will be that set which yields the smallest quadratic form

$$z'_F \hat{A}_{ip} z_F.$$

This will be inferior to the full-information-maximum-likelihood estimates. Another way of using limited-information estimates in practical forecasting, is by solving algebraically for reduced form equations in a structural system, each equation of which has been estimated by the method of limited information.[12])

The foregoing results on the efficiency of the use of *a priori* information follow closely a development by Samuelson called "the generalized Le Chatelier principle."[13]) Samuelson shows that the sensitivity of an economic variable to a parameter change, e.g., price elasticity of demand, decreases as the number of restraints imposed upon the system increases. He proves this proposition by bordering a matrix of second derivatives of a function being maximized. In a sense, this paper extends his

[12]) L. R. Klein and A. S. Goldberger, *An Econometric Model of the United States*, 1929—1952 (Amsterdam: North-Holland Publishing Co., 1955).

[13]) P. A. Samuelson, *Foundations of Economic Analysis* (Cambridge: Harvard University Press, 1947), pp. 36—39).

result from inequalities on diagonal elements of inverse bordered matrices to quadratic forms of principal minors.[14]) If we were interested solely in variances of individual coefficients in the reduced form, his result would be directly applicable. Since the forecast error involves an entire quadratic form, his proposition must be extended. It may also be remarked that Samuelson's proposition is not a perfect analogue for ours. His bordered matrices are those familiar in establishing conditions for extremes, while ours would have to be bordered further to take that form.

Some Experimental Sampling Results

The argument so far has been based on asymptotic theory. In a constructed experiment with repeated small samples, G. W. Ladd has produced some extremely interesting findings relevant to the problems of this paper.[15]) His model consists of two overidentified (demand and supply) equations.

$$(23)\qquad \begin{aligned} y_{1t} &= \beta_{12} y_{2t} + \gamma_{11} z_{1t} + \gamma_{12} z_{2t} + u_{1t} \\ y_{1t} &= \beta_{22} y_{2t} + \gamma_{23} z_{3t} + \gamma_{24} z_{4t} + u_{2t}. \end{aligned}$$

He assigned specific population values to the structural coefficients and drew random numbers for the disturbances and the exogenous variables. The endogenous variables, y_{1t} and y_{2t}, were then computed from the structural equations in (23). Ladd was particularly interested in the effects of errors of observation; therefore he added to each of the endogenous and exogenous variables an error of observation.

Thirty samples of thirty observations each were obtained by drawing the six sets of observation errors repeatedly, while the disturbances u_{it}, are not changed from sample to sample. The errors in y_{1t} are, together with linear functions of u_{it}, values of reduced form disturbances; thus the model differs from that treated above by having observation errors in the other variables. For each sample, Ladd computed from the observations on the

[14]) Samuelson's demonstration in the first printing contains some minor compensating errors, but the final results are quite correct.

[15]) G. W. Ladd, "Effects of Shocks and Errors in Estimation: An Empirical Comparison," *Journal of Farm Economics*, XXXVIII (1956), pp. 485—95.

y's and z's, estimates of the structural parameters by the method of least squares and the method of limited information.[16]) From each of these two types of estimates we can derive, algebraically, reduced form estimates in each sample.

If we denote least-squares estimates by a $\sim$ and limited information estimates by a $\wedge$ sign, we can derive two possible reduced forms for each sample,

$$\begin{aligned} y_{1t} &= \tilde{\gamma}_{11}\left(1+\frac{\tilde{\beta}_{12}}{\tilde{\beta}_{22}-\tilde{\beta}_{12}}\right)z_{1t}+\tilde{\gamma}_{12}\left(1+\frac{\tilde{\beta}_{12}}{\tilde{\beta}_{22}-\tilde{\beta}_{12}}\right)z_{2t} \\ &\quad -\tilde{\gamma}_{23}\frac{\tilde{\beta}_{12}}{\tilde{\beta}_{22}-\tilde{\beta}_{12}}z_{3t}-\tilde{\gamma}_{24}\frac{\tilde{\beta}_{12}}{\tilde{\beta}_{22}-\tilde{\beta}_{12}}z_{4t} \\ y_{2t} &= \frac{\tilde{\gamma}_{11}}{\tilde{\beta}_{22}-\tilde{\beta}_{12}}z_{1t}+\frac{\tilde{\gamma}_{12}}{\tilde{\beta}_{22}-\tilde{\beta}_{12}}z_{2t}-\frac{\tilde{\gamma}_{23}}{\tilde{\beta}_{22}-\tilde{\beta}_{12}}z_{3t}-\frac{\tilde{\gamma}_{24}}{\tilde{\beta}_{22}-\tilde{\beta}_{12}}z_{4t} \end{aligned} \tag{24}$$

or the same equations with $\tilde{\gamma}_{11}$ replaced by $\hat{\gamma}_{11}$, $\tilde{\gamma}_{12}$ by $\hat{\gamma}_{12}$, etc.

Alternatively, without taking any of the identifying restrictions into account, we can postulate general reduced form equations

$$\begin{aligned} y_{1t} &= p_{11}z_{1t}+p_{12}z_{2t}+p_{13}z_{3t}+p_{14}z_{4t} \\ y_{2t} &= p_{21}z_{1t}+p_{22}z_{2t}+p_{23}z_{3t}+p_{24}z_{4t} \end{aligned} \tag{25}$$

whose coefficients are estimated from the least-squares regressions of y_{1t} and y_{2t} on z_{1t}, z_{2t}, z_{3t} and z_{4t}. These regressions are obtained as a by-product of the limited-information estimates.

The accompanying table gives the results of our computations from Ladd's data. For the three methods of estimating the reduced forms, I have calculated the mean and standard deviation of the thirty sample values of each parameter. The computations in the table assume, in effect, that there are thirty independent samples; however, the fact that the vectors $(u_{i1}, u_{i2}, \ldots u_{i,30})$ are unchanged in repeated samples, means that the composite reduced form disturbance, consisting of a linear function of u_{it} and observation error, contains a common element in each sample. Since the variance of this linear function of u_{it} is not small relative to the variance of observation error there is significant correlation in the random terms from sample

[16]) In the least-squares regressions, y_{1t} is selected as the dependent variable.

to sample. In comparing the different methods of estimation we do not have as much information as would be given by thirty independent samples.

TABLE

Sample Means and Standard Deviations of Estimates of Reduced Form Parameters Obtained by Three Methods

1. Reduced form coefficients derived from least-squares estimates of structural parameters.

	Population parameter	Sample mean	Sample standard deviation
p_{11}	.05	.011	.017
p_{12}	.27	.094	.083
p_{13}	.06	.116	.040
p_{14}	.11	.127	.036
p_{21}	−.16	−.295	.280
p_{22}	−.90	−2.054	.360
p_{23}	.30	.670	.224
p_{24}	.56	.770	.323

2. Reduced form coefficients derived from limited-information estimates of structural parameters.

	Population parameter	Sample mean	Sample standard deviation
p_{11}	.05	.045	.046
p_{12}	.27	.254	.042
p_{13}	.06	.037	.026
p_{14}	.11	.123	.046
p_{21}	−.16	−.127	.118
p_{22}	−.90	−.957	.132
p_{23}	.30	.175	.089
p_{24}	.56	.559	.129

3. Reduced form coefficients estimated directly from least-squares regressions of y_{1t} and y_{2t} on z_{1t} z_{2t}, z_{3t}, and z_{4t}.

	Population parameter	Sample mean	Sample standard deviation
p_{11}	.05	.028	.094
p_{12}	.27	.258	.049
p_{13}	.06	.044	.071
p_{14}	.11	.121	.048
p_{21}	−.16	−.027	.314
p_{22}	−.90	−.951	.138
p_{23}	.30	.091	.242
p_{24}	.56	.576	.151

In comparing the first two methods, least-squares and limited information structural estimates, the bias of the former is clearly

shown.[16]) In all cases, the mean of the limited-information estimate is closer than the mean of the least-squares estimate to the population value. In six of eight cases, the sample standard deviation of parameter estimates is lower for limited-information than for least-squares estimates. This is a reflection of the proposition that, in general, the optimal properties of least-squares estimates are not preserved under transformation.[17])

In comparing the last two methods, it can be seen that neither shows much bias, which is as expected, but that limited-information estimates have uniformly smaller variance. This is a small sample exposition of the results derived in the preceding section.

Ladd's results are extremely suggestive for our purposes though not conclusive. Of course, results from constructed experiments do not constitute general proof of investigated propositions, but Ladd's experiment is far less than perfectly designed for the particular objectives of this paper. It would be desirable to carry through similar experiments without the complication of observation error. Even within the framework of Ladd's experimental design, there are changes that could be made to strengthen the present inquiry. Both least-square structural regressions were made with y_{1t} as the dependent variable, but other choices deserve to be investigated in the present context. This is especially true since the population correlation between u_{2t} and y_{2t} is as high as 0.56. It is the correlation between disturbances and independent variables in regression analysis that introduces bias. The results in the table are also restrictive in that they deal only with variances and neglect covariance. A single statistic constructed along the lines of the standard error of forecast would be more adequate. In a future sampling inquiry some of these deficiencies can be remedied.

[16]) Limited information estimates are consistent but not generally unbiased. In small samples of the type under consideration, bias may be revealed in them as well. Here the argument is simply that limited information estimates reveal, in the sampling experiment, a smaller degree of bias.

[17]) The standard deviations are computed about sample means. If they were computed about population means, the least-squares results would compare even less favorably.

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